An update to the history of peg solitaire

John Beasley, 10 August 2014, appendices added to 17 April 2017
(*a paper for the International Puzzle Party, revised in the light of feedback received*)

This paper has two objectives: to present some recent discoveries, mainly by Dic Sonneveld, regarding the history of "standard" peg solitaire, and to look at some of the interesting non-standard boards that have been used over the years.

There still seem to be two standard modern references for peg soltaire: chapter 23 of the classic *Winning Ways for your Mathematical Plays* by E. L. Berlekamp, J. H. Conway, and R. K. Guy (Academic Press, 1982), and my own *The Ins and Outs of Peg Solitaire* (OUP, 1985, paperback edition with minor updates 1992). If any reader knows of anything which supersedes these, please will he or she bring it to my attention. We shall also refer to the Peg Solitaire page of www.jsbeasley.co.uk, where I posted "Contributions towards a historical update" last year. *Winning Ways* went into a second edition (A. and K. Peters, 2001-04), and as far as I know this is still in print. *The Ins and Outs* is out of print and is likely to remain so, but copies should be available in academic and other major libraries.

The classical 37-hole board

The first board to have been regularly used for peg solitaire appears to have been the classical 37-hole "French" board

		1	**2**	**3**		
	4	**5**	**6**	**7**	**8**	
9	**10**	**11**	**12**	**13**	**14**	**15**
16	**17**	**18**	**19**	**20**	**21**	**22**
23	**24**	**25**	**26**	**27**	**28**	**29**
	30	**31**	**32**	**33**	**34**	
		35	**36**	**37**		

where moves are permitted in any horizontal or vertical direction but not diagonally. This appears first to have been mentioned in print in letters in the issues of *Mercure Galant* for August and September 1697, recently drawn to my attention by Dic Sonneveld (he credits knowledge of the August letters to French Wikipedia, but appears to have dug up the September letter himself). These are available on the web (I have posted references on the Peg Solitaire page of www.jsbeasley.co.uk), but the format is not particularly convenient, and for the convenience of interested readers I have also posted a transcription as a PDF.

The first of these letters credits the invention of the game to the North American Indians, who allegedly played it with arrows stuck into the ground on returning from a hunt. This legend has been widely quoted, but the writer was rather tentative about the matter ("I am told that ... it may be that ... whatever country this Foreigner may have come from", my translations) and I believe the truth to have been much more mundane: a board with this pattern of holes was already in use for some other purpose, and somebody noticed that it made a good puzzle.

The natural challenge is to start with a board full apart from a single hole and play so as to reduce to a single peg, but this is not always possible. In particular, if the initial vacancy is in the centre then no single-peg finish is possible (we shall prove this in the next section). What *is* possible is to start by vacating hole **1**, **18**, **21**, or **35**, and play to leave a peg in **3**, **17**, **20**, or **37**, and to do the equivalent from similar vacancies elsewhere (for example, we can start by vacating **9**, **12**, **15**, or **32**, and play to leave a peg in **6**, **23**, **26**, or **29**). Counting positions equivalent under rotation or reflection as the same, this gives ten different solvable single-vacancy-to-single-survivor problems. Solutions to several are in the letters to *Mercure Galant*, including one to "vacate **3**, play to finish at **35**" in which the peg at **2** sweeps off nine men in succession before being itself jumped over.

This last solution, and some others, are in an engraving "Nouveau Jeu du Solitaire" by Berey, and there are also extant engravings by Berey and his contemporary Trouvain showing one of Louis XIV's mistresses and a "lady of quality" playing solitaire. (In those days, "solitaire" unambiguously meant the peg-board game, not a card game as in modern America, and a contrary statement by myself in *The Ins and Outs* was apparently quite misguided.) These engravings were originally undated (the dates "1697" and "1698" appearing on two of them as frequently reproduced seem to have been manuscript additions to printed copies), but it is reasonable to assume them to be roughly contemporary with the letters to *Mercure Galant*, and the recorded periods of activity of Berey and Trouvain are consistent with this. It would therefore appear that solitaire was very much the "Rubik's cube" of the court of Louis XIV.

The lady in one of the Berey engravings is using a board featuring a short handle, and another contemporary item showing what appears to be such a board has been drawn to my attention by Dic Sonneveld. This is in a volume containing part of a census of French armorial bearings taken under the direction of Charles d'Hozier, "juge d'armes du royaume de France", whose clerks registered no fewer than 110,000 shields between 1696 and 1709. This particular volume appears to have been signed off in 1711, and page 718 shows a shield, annotated "Jaque Chavillot, prètre habitué en l'Eglise Catédrale de S^{t}. Vincent de Chalon", featuring what appears to be a 37-hole solitaire board complete with handle. It is of course possible that there is no connection, but on the face of it this is further evidence that solitaire was sufficiently well established by 1709 for someone to have chosen its board as the symbol on his coat of arms. Those interested will find the relevant material at

http://gallica.bnf.fr/ark:/12148/bpt6k1111382/f2.item
http://gallica.bnf.fr/ark:/12148/bpt6k1111382/f721.item

(title page of the volume and page 718 respectively).

Although the "centre-peg game" (vacate the central hole and play to finish there) cannot be solved on the classical 37-hole board, many other attractive problems, both from a central-vacancy start and from other starting positions, can be set and solved on it. I have already mentioned the problems which were set and solved in *Mercure Galant* in 1697. Dic Sonneveld has drawn my attention to a book by de Bouis, *Le Nouveau Jeu du Solitaire, réduit en problêmes géometriques et en décorations enluminées*, Paris, 1753, which is described in Appendix 1, and the *Neueste Anweisung* of 1807, which we shall meet when we look at the 33-hole board, also covers the 37-hole board and gives some problems with a central-vacancy start and a target position with four-fold rotational symmetry. There was a further flurry of material in the late nineteenth century: P. Busschop's book of 1877, A. Deveau-Carlier's of 1884-85 (this went into at least three editions, only the first of which is mentioned in my bibliography in *The Ins and Outs*), Paul Redon's of 1893, and some problems by A. Huber and others which appeared in *Les Tablettes du Chercheur* (a French games-and-puzzles magazine) between 1891 and 1895. *Les Tablettes* and the later editions of Deveau-Carlier's book are yet more of the items which have been brought to my attention by Dic Sonneveld.

Conspicuous by its absence is any early reference to the legend that solitaire was invented by a prisoner in the Bastille. I wrote in *The Ins and Outs* that the earliest reference to this that I had seen was in Strutt's *Sports and Pastimes of the People of England* of 1801, which is of course quite worthless as evidence for an event which is alleged to have happened in a foreign country over a century before, and nobody has yet brought an earlier reference to my attention. I therefore stand by what I wrote some years ago in *The Games and Puzzles Journal*: anyone who repeats this tale without citing a French source earlier than 1801 should regard himself as perpetuating myth rather than history.

The proof that the centre-peg game on the 37-hole board cannot be solved

It was realised almost from the start that the centre-peg game on the classical 37-hole board could not be solved. To quote from one of the letters to *Mercure Galant*, "People have tried to play by starting at the Central Point and removing this man from the board, and also to finish the Game at the Central Point, but nobody has yet found a method of doing these, even though they have given them as much attention as to the Squaring of the Circle" (again my translation). However, it seems to have taken a surprisingly long time for a proof to be published. The *Neueste Anweisung* of 1807 appears to contain some problems, without solutions, which would surely not have been included had its author been aware of the proof outlined below, and Dic Sonneveld has drawn my attention to a report in the *Procès-verbaux des séances de l'Académie (Académie des sciences)* for 1 June 1818, by Arago and Cauchy, which suggests that it was still not generally known as late as this. However, Dic also draws my attention to an article in *Revue de la Côte d'Or et de l'ancienne Bourgogne*, 1836, pages 45-58, which gives a report on the work of Suremain de Missery earlier than that which I cited in *The Ins and Outs*. Sadly, de Missery's own *Traité analytique du jeu de Solitaire* appears to have been lost.

All this is curious, because a proof, once suspected, is very easy to construct, and one has been independently rediscovered many times. Suppose that we letter the holes in a row *A*, *B*, *C*, *A*, *B, C* ... in order, that we do the same with the next row down putting a *B* underneath each *A* in the first row, the same again with a third row putting a *C* beneath each *B* in the second row, and so on as in the diagram on the next page:

		A	*B*	*C*		
	A	*B*	*C*	*A*	*B*	
A	*B*	*C*	*A*	*B*	*C*	*A*
B	*C*	*A*	*B*	*C*	*A*	*B*
C	*A*	*B*	*C*	*A*	*B*	*C*
	B	*C*	*A*	*B*	*C*	
		A	*B*	*C*		

Having done this, we see that each line of three, horizontal or vertical, contains precisely one *A*, one *B*, and one *C*, so a jump alters the numbers of men in *A*, *B*, and *C* holes each by 1 (it increases one of them and decreases the other two). Now if we leave the centre vacant and count up the numbers of men in the other holes, we find that we have twelve men in *A* holes, twelve in *B* holes, and twelve in *C* holes, and each of these three numbers is even. Make a jump and they all change by 1, so they all become odd; make a second jump, and they all become even again; and so on. If we make an odd number of jumps, they all become odd; if we make an even number of jumps, they all become even; *they always remain of the same parity as each other*. But the intended target position has one *B* peg (odd) and no *A* or *C* pegs (both even), which are not all of the same parity, so the problem cannot be solved, and a similar argument applies if the final survivor is to be left in any other hole.

Are we really expected to believe that nobody sat down between 1697 and 1836, and worked this out for himself?

The 45-hole board

The next board to be used seems to have been the 45-hole board shown below:

			1	**2**	**3**			
			4	**5**	**6**			
			7	**8**	**9**			
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
19	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**
28	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**
			37	**38**	**39**			
			40	**41**	**42**			
			43	**44**	**45**			

Dic Sonneveld draws my attention to a mention of a "Capuzinerspiel" using this number of pegs on page 54 of *Das dreyseitige Schachbrett, oder, Art und weise: auf demselben sich Selbdritte zu unterhalten, aus dem Italiänischen mit verschiedene Hauptsätzen*, 1765 (apparently primarily a book on a three-handed version of chess invented by Filippo Marinelli in 1722), and a board with this particular shape was mentioned by J.-C. Wiegleb in his *Unterricht in der Natürlichen Magie* of 1779. Wiegleb gives a solution to the problem "vacate **1**, finish at **13**", with a note that if the initial vacancy is at any hole other than **2**, **14**, **20**, and **32** it is possible to leave the final peg in the hole which was initially vacant. This statement is not correct (the correct list of exceptions is **2**/**19**/**27**/**44**), but he may have had in mind the notation for the 33-hole board which we shall meet in a moment. Even so, if he did indeed have solutions to all these "vacate **X** and finish there" problems, it is surprising that he gives one only to the relatively easy and uninteresting "vacate **1**, finish at **13**".

This 45-hole board appears to have fallen out of use fairly soon, though Dic tells me that the *Oberdeutsche* *allgemeine Literaturzeitung* of 1804 mentions that the "Einsiedler- oder Kapuzinerspiel" usually had 44 pegs, and that an article "De Puzzle van Napoleon", in the issue of *Algemeen Handelsblad* for as late a date as 20 October 1928, again shows the 45-hole board. However, when faced with an isolated and anomalous item such as this last, one always wonders whether a non-playing writer has got the details wrong. The point is addressed further in Appendix 2.

Play on the 45-hole board is relatively difficult, and no further examples seem to have appeared in print until I blocked out solutions to "vacate and finish at **1**" and "vacate and finish at **5**" in *The Ins and Outs*. If it is wondered why I did not give a solution to the centre-peg problem "vacate and finish at **23**", the answer is that one follows immediately from that to the more difficult problem "vacate and finish at **5**" (the first and last jumps of the latter must be **14-5**, and if we vacate **23** instead and replace these first and last jumps **14-5** by **8-23** we have a solution to "vacate and finish at **23**"). More recently, George Bell and his computer have been looking at the game, and some of his discoveries are summarized in a paper "New problems on old solitaire boards" which we wrote for a board games colloquium in Oxford in 2005 (now published in volume 8 of *Board Game Studies Journal*, pages 123-145, available on-line at http://bgsj.ludus-opuscula.org). The board was worth revisiting.

The German tradition appears to have been that peg solitaire was invented by monks or nuns to pass away their time in religious seclusion, and around 1800 the names Einsiedlerspiel, Kapuzinerspiel, Kreuzspiel, and Nonnenspiel were in use for the game. The cross shape of the 45-hole board, and of the 33-hole "German solitaire" board which we shall meet in the next section, has been held to add plausibility to this. I have to say that I am as suspicious of this as I am of all picturesque legends about the invention of games, but that versions of the game evolved independently in France and in Germany is by no means impossible.

The modern 33-hole board

The modern "German" or "English" 33-hole board

		1	**2**	**3**		
		4	**5**	**6**		
7	**8**	**9**	**10**	**11**	**12**	**13**
14	**15**	**16**	**17**	**18**	**19**	**20**
21	**22**	**23**	**24**	**25**	**26**	**27**
		28	**29**	**30**		
		31	**32**	**33**		

appears first to have been mentioned by Wiegleb in 1779. He gives a solution to the problem "vacate **1** and leave a single survivor there", but goes on to say that if the central hole is the one left vacant at the start then the final peg may be left in any hole other than **9**, **11**, **23**, or **25**, which is nonsense. In fact a "rule of three" applies: the final survivor may be left in the hole initially vacant, or in any hole a multiple of three away in any direction (so if the initial vacancy is at hole **1**, the final survivor may be left in hole **1** itself or in hole **16**, **19**, or **31**). That no other case is possible can be proved by an *ABC* argument such as proves the unsolvability of "vacate and finish at **19**" on the 37-hole board.

I have seen one or two German advertisements from the years around 1800, but these contain no examples of play, and the next solutions of which I am aware are in an anonymous 1807 book *Neueste Anweisung zum Kreuz-, Einsiedler- oder Kapuzinerspiel* which is in my bibliography in *The Ins and Outs* only as "cited by Ahrens". Thanks to Dic Sonneveld, I have now seen this, and it contains diagrams and solutions covering nearly all the positions with eightfold symmetry and four-fold rotational symmetry which can be reached from a central-vacancy start. They include the centre-peg finish.

Also brought to my attention by Dic is a similarly anonymous 1808 book *Der praktische Solitärspieler oder Anweisung das bekannte Kreuz- oder Kapuzinerspiel durch afgestellte Muster zu erlernen*. This is more difficult for me to assess, because I do not read German and the copy I have seen appears to lack diagrams, but in so far as I can judge it contains a complete set of single-vacancy single-survivor problems subject to the "rule of three" mentioned above. Furthermore, most of the problems are of the "marked man" kind where a specific peg is nominated to be the final survivor. These are presented in sets where the nominated survivor is required to make the *n*th move of the solution, *n* running from 1 to 7, but there appears to be no highlighting of "man on the watch" problems (where the marked man remains motionless until the final sweep) and no attempt is made to maximize the length of this final sweep. If the initial vacancy is at **2** and the marked man is at **10**, the marked man always finishes at **32**, but being unable to read the text I cannot say whether the author claimed to have proved this to be the only solvable case or whether he was merely confining his presentation to problems to which he had been able to find an answer. When I was writing *The Ins and Outs*, friends translated for me.

The subsequent literature has been vast, and even my bibliography in *The Ins and Outs* had to be highly selective. Here, we can do no more than give a few highlights. Reiss's frequently cited paper of 1857 is now seen to contain only material already published elsewhere, but it was produced in good faith and I remember its treatment of single-vacancy single-survivor problems as having been much more simple and systematic than that in *Der praktische Solitärspieler*. C. Bizalion gave many interesting problems in articles in the *Gentleman's Journal Recreation Supplement* between 1870 and 1872, and in particular seems to have pioneered "man on the watch" problems with a long final sweep. More such problems were given by Ernest Bergholt in his 1920 *Complete Handbook to the Game of Solitaire on the English Board of Thirty-Three Holes*. Attention was also given to minimising the number of separate moves, counting a series of jumps by the same man as a single move; in 1912, Bergholt published an 18-move solution to the centre-peg game, and in 1967 Harry O. Davis published a 15-move solution to the problem "vacate and finish at **9**". Both are now proven minima.

And in *Winning Ways*, John Conway reported the theoretical work which he had done with J. M. Boardman and R. L. Hutchings in 1961-63. This has proved invaluable in resolving problems which are unsolvable but had previously been difficult to prove so, and in identifying which of a set of candidate jumps must be used to empty a particular hole if an unsolvability is not to result. In particular, it gave a formal proof that if the hole at **2** is initially vacated and the man at **10** is nominated to be the final survivor, this man must finish far away at **32**.

Other boards with a square lattice

The **41-hole diamond board**

				1				
			2	**3**	**4**			
		5	**6**	**7**	**8**	**9**		
	10	**11**	**12**	**13**	**14**	**15**	**16**	
17	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
	26	**27**	**28**	**29**	**30**	**31**	**32**	
		33	**34**	**35**	**36**	**37**		
			38	**39**	**40**			
				41				

appears to have been first described by Édouard Lucas in his *Récréations Mathématiques* of 1882, citing work performed a few years earlier by H.-A.-H. Hermary. The *ABC* theory, which applies to this board just as it does to all boards using a square lattice, suggests that while the centre-peg problem "vacate and play to finish at **21**" cannot be solved, we may hope, by analogy with the 37-hole board, to be able to start by vacating **2**, **17**, **20**, **23**, or **38**, and to finish in **4**, **19**, **22**, **25**, or **40**. In fact we cannot. Suppose we fill the holes with white and black pegs alternately, chequerboard style, with black pegs around the outside. For a full board, this requires 25 black pegs against only 16 white pegs, and each move of an edge or corner peg into the interior of the board to be captured will remove a white peg. So a problem like "vacate **2**, finish at **22**" is seen almost at once to be unsolvable. The first move removes a white peg, and even if it is **4-2** it still leaves 15 outside pegs to be jumped into the interior of the board to be captured; each of these necessary jumps will remove a further white peg; so no white peg will be left to occupy **22** at the end.

Arguments of this kind reduce us to five candidate problems: "vacate **2** or **23**, finish at **4** or **19**", and "vacate **2**, finish at **40**". "Vacate **23**, finish at **19**" was solved by Hermary, and quoted by Lucas in 1882; the complete quartet "vacate **2** or **23**, finish at **4** or **19**" was solved as a set by Paul Redon in 1888, and the solutions quoted by Lucas in his second edition in 1891; "vacate **2**, finish at **40**" was proved unsolvable by Redon in the issue of *Les Tablettes du Chercheur* for 1 May 1892.

There have been several derivatives of the 41-hole board, and in particular Hermary removed holes **17** and **25**, giving **Hermary's 39-hole board**. This has only rectangular symmetry, but on it the centre-peg problem can be solved. Indeed, all single-vacancy single-survivor problems which satisfy the "rule of three" can be solved apart from "vacate **3**, finish at **39**" and "vacate and finish at **18**". Redon proved these last two to be unsolvable in the issues of *Les Tablettes* for 15 March and 1 April 1892.

According to his obituary in the issue of *Les Tablettes* for 1 July 1895, Redon died at the age of 32, having been suffering from tuberculosis since he was 18, and his mother had introduced him to peg solitaire as a comforter while he was confined to bed. His proofs of unsolvability mentioned above were analyses of the highest class, and anticipated the techniques which were to be developed in a more general form by Conway and Hutchings in 1961 and crystallized by Conway in his "balance sheet". I know of no evidence that he analysed more than these three cases, but this was more than enough to give him an honoured place in the history of the game. I would certainly have acknowledged his work in *The Ins and Outs* had I been aware of it at the time.

The **6 x 6 square board**

1	**2**	**3**	**4**	**5**	**6**
7	**8**	**9**	**10**	**11**	**12**
13	**14**	**15**	**16**	**17**	**18**
19	**20**	**21**	**22**	**23**	**24**
25	**26**	**27**	**28**	**29**	**30**
31	**32**	**33**	**34**	**35**	**36**

appears to have received little attention before Martin Gardner discussed it briefly in his *Scientific American* column in 1962. This is the smallest board on which it is possible both to select any initial vacancy and to require any particular peg to be the final survivor (the "rule of three" says that the survivor must finish either in the initial vacancy or in a hole a multiple of three away in each direction, and the fact that each jump moves a peg two holes means that the surviving peg must finish either in its starting position or in a hole a multiple of two away in each direction). Indeed, we can do more; we can require this peg to remain motionless until it makes the final sweep, and this sweep can be made always to remove at least eight men and in some cases to remove as many as ten. Wade E. Philpott exploited this in a game **10-Leap Solitaire** which was marketed by Kadon Enterprises in 1982.

If we regard successive jumps by the same peg as a single move, it is easily proved that any single-vacancy single-survivor problem must take at least 15 moves, and at least 16 if the initial vacancy is in a corner. In 1962, probably in response to the mention of this board by Martin Gardner, John W. Harris found a 16-move solution to the problem "vacate and finish at **1**", and Harry O. Davis subsequently found 16-move solutions to the companion problems "vacate **1**, finish at **4** or **22**". Davis also found two single-vacancy single-survivor problems which allowed 15-move solutions, Harris added a third by hand, and Harris's computer then resolved the rest; it found a 15-move solution to every problem apart from "vacate and finish at **3**", and proved this last problem to require at least 16 moves.*

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* I reported Harris's work in 2003 in issue 28 of *The Games and Puzzles Journal*, on the grounds that it had been done seventeen years or more before, that other people were beginning to reproduce his results, and that I ought to report what I knew if only to establish his priority. However, when I had temporary custody of David Pritchard's chess variant papers following his death, I found that they included a complete run of Michael Keller's *World Game Review*, and although I took no copy I remembered one of these as having contained an article by Harris. I presumed this had contained his own report of the work in question, and I therefore wrote in various places that my report in 2003 appeared to have been unnecessary. In fact this seems not to have been entirely the case. David Singmaster has searched issues 1-12 of *World Game Review* for me (he tells me he does not have convenient access to issue 13), and he has sent me copies of the item in question and of some further solitaire material by Harris and by Leonard Gordon which is referred to later in this paper. It now appears that the item (in issue 8, July 1988, page 2) was not a full article but merely a brief letter giving much less detail than subsequently appeared in *The Games and Puzzles Journal*, and the two are therefore complementary.

David Pritchard's copies of *World Game Review* are now in the Musée Suisse du Jeu, having been forwarded in 2012 with the rest of his chess variant papers (see the Chess Variants page of www.jsbeasley.co.uk).

The 33-hole and 45-hole boards can be regarded as 3 x 3 square boards with arms of length 2 or 3 on each side. In the early years of this century, George Bell looked at **generalized cross boards**, in which these arms can be of any lengths, and found that his computer took a long time to solve the problem “vacate and finish at **2**” on a 39-hole board

		1	**2**	**3**		
		4	**5**	**6**		
		7	**8**	**9**		
10	**11**	**12**	**13**	**14**	**15**	**16**
17	**18**	**19**	**20**	**21**	**22**	**23**
24	**25**	**26**	**27**	**28**	**29**	**30**
		31	**32**	**33**		
		34	**35**	**36**		
		37	**38**	**39**		

which was a half-way house between the 33-hole and 45-hole boards. Subsequent analysis showed the solution of this problem to be unique to within symmetry and ordering of jumps. Several single-vacancy single-survivor problems on very small boards have this property, but that one should have it on a naturally-shaped board as large as this was wholly unexpected.

George’s work on generalised cross boards is reported in issue 28 of *The Games and Puzzles Journal* (available on-line on George Jelliss’s “Mayhematics” site www.mayhematics.com), and a proof that the solution to the problem above is unique to within symmetry and ordering of jumps can be found in our paper for the 2005 board games colloquium.

Many other square lattice boards have been tried out over the years, on paper if not necessarily in wood. A. Huber removed holes **5**, **9**, **33**, and **37** from the 41-hole diamond board, giving **Huber’s 37-hole board** on which the centre-peg problem can be solved. He presented some problems on this board in *Les Tablettes du Chercheur* in 1894 and 1895, but it seems to have received little further attention until Leonard Gordon and John Harris looked at it in 1988 (see issue 8 of *World Game Review*, pages 21-23). As well as the centre-peg problem, most single-vacancy single-survivor problems subject to the normal “rule of three” can be solved, but the “apex” problem “vacate and finish at **1**” cannot; indeed, if we start by vacating **1**, a single survivor can finish only in **10** or the symmetrically equivalent hole **16**.

Another board considered in *Les Tablettes* was the **32-hole draughtsboard**, some problems on which were presented by E. Furundarena in 1895 and 1896. Here, the “apex” problem can be solved, but the centre-peg problem cannot even be posed because there is no central hole.

On the theoretical side, around 2003 George Bell and I looked at **long-arm boards**, where three-hole-wide arms of various lengths are tacked on to the sides of a square or other simple figure. The most difficult problem is always the equivalent of “vacate and finish at **2**” on the 33-hole and 45-hole boards, where the initial vacancy is in the middle of one end of an arm. This problem is solvable on the 129-hole board obtained by attaching four 4 x 3 arms to a 9 x 9 square, and to our surprise it turned out to be solvable also on the 285-hole board obtained by attaching four 5 x 3 arms to a 15 x 15 square, but it is never solvable on a board with a 6 x 3 or longer arm. This also was reported in issue 28 of *The Games and Puzzles Journal*.

And even a **semi-infinite board** has been considered. Back in 1961 or 1962, Conway asked the question, “If we start with all our men behind a horizontal line, how many men does it take to send a scout forward *N* ranks?” One rank, two men (trivial); two ranks, four men (almost as trivial); three ranks, eight men (easy); four ranks, twenty men (more difficult); five ranks? Surprisingly, it cannot be done, however many men we have. However, it is “only just” impossible; if we allow ourselves to start with two men in one of the holes, the task becomes possible no matter how far back this hole may be.

The proof of this impossibility is one of the most elegant pieces of Solitaire mathematics.

Boards with a triangular lattice

Boards not on a square lattice receive fairly short shrift in *The Ins and Outs*, and no shrift at all in *Winning Ways*. Let us try and do a little better.

Triangular boards have an unfortunate property. If the board is of side $3n + 1$, say

1

2 3

4 5 6

7 8 9 10

the centre-peg problem cannot be solved (proof for $n > 1$ by an *ABC* argument similar to that which we used for the classical 37-hole board); if the board is of side $3n + 2$ or $3n + 3$, say

1

2 3

4 5 6

7 8 9 10

11 12 13 14 15

1

2 3

4 5 6

7 8 9 10

11 12 13 14 15

16 17 18 19 20 21

there isn't even a central hole. However, in the latter two cases (sides $3n + 2$ and $3n + 3$), the "apex" problem "vacate and finish at **1**" can be solved. If the board is of side $3n + 1$, even the apex problem cannot be solved. (In all our formulae of this kind, n may be any integer from 1 upwards.)

All this being said, these boards can be quite difficult to play on, and the apex problem provides a substitute for a solvable centre-peg problem. David Singmaster has sent me a photocopy of a 15-hole board made in England as one of a series of puzzles for sale through restaurants, and Seph Barker and George Bell tell me that similar boards used to be and perhaps still are provided at tables in certain restaurant chains in America. They have also stimulated some interesting theoretical work, in particular a paper "Triangular Puzzle Peg" by I. R. and R. R. Hentzel in volume 18 issue 4 (1985-86) of *Journal of Recreational Mathematics*, pages 253-256.

A **hexagonal board** always has a central hole, but the centre-peg problem can be solved only if the board is of side $3n + 2$, and the smallest such board is the 61-hole board of side 5. This has not prevented 37-hole boards of side 4 from being marketed, sometimes with challenges to the player (either through ignorance or through malice) to do what is in fact impossible.

A **stellar board** can be obtained by taking two triangular boards of side $3n + 1$, reflecting one top to bottom about the line through its central hole, and superimposing them:

1

2 3 4 5

6 7 8

9 10 11 12

13

Unfortunately the centre-peg problem can be solved only if the triangles are of side $9n + 4$, and the smallest such board has no fewer than 121 holes. But the apex problem can be solved on the 13-hole board above, as can attractive problems such as "vacate **1**, mark the peg at **13**, and play to leave this peg back at **13** having taken four men in its final sweep" and "vacate **1**, mark the pegs at **6** and **8**, and interchange them clearing everything else".

However, there are triangular lattice boards of reasonable size on which the centre-peg problem can be solved. One such, sent to me by Jerry Slocum when I was working on *The Ins and Outs*, is **Smith's 16-hole board**

1 2 3

4 5

6 7 8 9 10

11 12 13 14

15 16

which may be regarded as a superposition of three 3-2-1 triangles with a common apex. The centre-peg problem can be solved on this board, though no other single-vacancy single-survivor problem can. Smith went as far as to patent the board (U. S. patent 462,170 of 1891), though whether this benefitted anyone other than his patent attorneys is perhaps an interesting question.

It is also possible to superimpose two triangles with a common side, producing a **diamond board**. On the face of it, diamond boards offer little more than is provided by ordinary square boards; as on a square board, the centre-peg problem can be solved only if the board is of side $6n + 3$, and even the apex problem can be solved only if the board is of side $3n + 3$. However, David Singmaster has drawn my attention to a 16-hole board marketed as **Touchdown**, in which the two central holes are designated a "central zone" and made a special object of play (in particular, it is possible to vacate one of them and play to leave a single survivor in the other):

1

2 3

4 5 6

7 8 9 10

11 12 13

14 15

16

David has also sent me an **hour-glass board**, in which the lengths of the rows decrease to a minimum and then increase again. In their strictest form, where the length of the rows decreases right down to 1 and then returns to its original value, hour-glass boards offer even less than diamond boards, because the centre-peg problem cannot be solved however many rows we have. The board which David sent me was of this type. However, enjoyable puzzles can be produced by allowing the "waist" to consist of three holes instead of one, say

1 2 3 4 5

6 7 8 9

10 11 12

13 14 15 16

17 18 19 20 21

or the same with seven rows instead of five. The centre-peg problem can be solved in each case.

These are only a few of the triangular lattice boards that can be used. George Bell has looked at equivalents of Smith's board with larger triangles, calling them **propellor boards** (sadly, the centre-peg problem can be solved only if the triangles are of side $3n$ or $3n + 2$, which excludes the conveniently-sized 28-hole board of side 4). More general boards with a central hole can be produced by the symmetrical removal of chunks from a hexagon, retaining hexagonal symmetry, triangular symmetry (as in Smith's board), rectangular symmetry (as in diamond and hour-glass boards), or even just rotational symmetry. We leave the exploration of such boards to the reader.

Square lattice boards with diagonal moves (see also Appendix 5)

The allowing of diagonal moves on square lattice boards opens up further possibilities. Consider the 13-hole diamond board

```
        1
    2   3   4
5   6   7   8   9
   10  11  12
       13
```

If we allow only horizontal and vertical moves, this is hopeless; the pegs in holes **2**, **4**, **10**, and **12** cannot be brought into the interior of the board to be captured. If diagonal moves are allowed, these pegs can be jumped over where they stand.

With the help of diagonal moves, the following single-vacancy single-survivor problems can be solved.

- Initial vacancy at **1**: the final survivor may be in any hole except **6** or **8**.
- Initial vacancy at **2**: the final survivor may be in any of the eight holes around the edge.
- Initial vacancy at **3**: the final survivor may only be in hole **1** or **13**.
- Initial vacancy at **7**: the final survivor may be in hole **7** itself or in any of the four corner holes.

So the centre-peg and apex problems can both be solved, and there are also interesting problems of the "swap" kind; for example, "vacate **7**, mark the pegs at **5** and **9**, and play to interchange them and clear the rest of the board" and "vacate **2**, mark the pegs at **1** and **5**, and play to interchange them similarly".

All this being said, the addition of diagonal moves does tend to make the game too easy. Even on the 4 x 4 square board, all single-vacancy single-survivor problems can be solved; we can start by vacating any hole, and leave the final survivor either in the same hole or in any other. The same is true of the 21-hole board obtained by removing the outer rows and columns from a standard 33-hole board, a form of the game which has been marketed. However, diagonal moves open up new possibilities for long-sweep and fewest-moves solutions. Leonard Gordon and John Harris have looked into these, and some of their results are reported in issue 11 of *World Game Review*, pages 17-18 (and much more briefly in issue 28 of *The Games and Puzzles Journal*).

Boards using arbitrary networks

In principle, peg solitaire can be played on any network of intersecting lines. George Bell has sent me a picture of **Star Jump**, a ten-hole wood-and-marbles puzzle in the form of a pentagram, which is very attractive although the play is easy; it is possible to vacate any of the five outside holes and play to leave a final survivor in any of the five inside holes, or vice versa (see also Appendix 4). And when I was writing *The Ins and Outs*, Martin Gardner suggested using the board for a game called "Solomon", producing a puzzle which it was natural to call **Solomon Solitaire** :

```
          1

2      3  4  5      6
    7          8
      9   10   11
    12         13
14     15 16 17    18

          19
```

Here, jumps are possible along any line containing five holes (so jumps such as **2-4**, **3-5**, **4-6** and **1-10**, **4-16**, **10-19** are permitted, but jumps such as **9-11** are not). The centre-peg problem on this board can be solved; this apart, the holes divide into three sets typified by **1**/**4**/**16**/**19**/**9**/**11** (in other words, the holes in two lines at right angles through the centre, but excluding the centre itself), and a single-vacancy single-survivor problem can be solved if and only if the initial vacancy and the target hole are in the same set.

Nor need the lines be straight. David Singmaster recently sent me a picture of a board manufactured a few years ago by Puffin Toys of Charmouth, Dorset, which was laid out approximately as

1		**2**		**3**		**4**
	5		**6**		**7**	
8						**9**
	10		**11**		**12**	
13						**14**
	15		**16**		**17**	
18		**19**		**20**		**21**

with a red peg in the centre and light yellow pegs everywhere else. There was no accompanying leaflet or set of rules, and it may not have been a puzzle at all but merely a toy for children. However, we have a board, so let's try to play **Puffin Solitaire** on it. If we allow only horizontal and vertical moves and moves along the long diagonals, no single-vacancy single-survivor problem is solvable. However, if we regard lines such as **2-6-12-14** as straight, which they nearly are, and allow moves such as **2-12** over **6**, all single-vacancy single-survivor problems are solvable. The same is true if we allow bifurcation, permitting moves such as **11-2** and **11-3** over **6**.

A game with some explicitly non-straight lines is **Round Solitaire**, invented in 2009 by Tetsuro Kawahara and brought to my attention by George Bell. This uses a 21-hole board

	12	**1**	**2**	
11	**13**	**14**	**15**	**3**
10	**16**	**17**	**18**	**4**
9	**19**	**20**	**21**	**5**
	8	**7**	**6**	

with the usual horizontal and vertical jumps plus quadrants linking holes **2-3**, **5-6**, **8-9**, and **11-12** so that **1-2-3-4-5-6-7-8-9-10-11-12-1** is a continuous ring and jumps such as **1-3** over **2** and **11-1** over **12** are possible. The centre-peg problem can be solved; more generally, the holes divide into two sets, the nine holes forming the cross through the centre and the remaining twelve, and a single-vacancy single-survivor problem can be solved if and only if the initial vacancy and the target hole are in the same set.

Further boards featuring non-straight lines are **Clock Solitaire** and the **Ancient Star Problem Puzzle**, which are described in Appendices 3/5 and 6, and

		1		
6		**7**		**2**
	12		**8**	
		13		
	11		**9**	
5		**10**		**3**
		4		

which was on sale in London around 1990 as **The Crystal Palace Wheel Puzzle**. I have seen only a photocopy of an eight-page descriptive booklet (A6 size) containing the rules and a set of problems with neither solutions nor acknowledgement, but there may have been a separate leaflet or booklet containing these. The puzzle was in fact suggested by myself and all the problems were supplied by myself, perhaps somebody else had already thought of it. There are rings **1-2-3-4-5-6-1** and **7-8-9-10-11-12-7**, and jumps are allowed along these and along the diameters. The centre-peg problem can be solved (we leave the resolution of other single-vacancy single-survivor problems to the reader), and there are also attractive problems in which the central hole **13** is vacated and the object is to exchange pegs **1** and **4**, or **7** and **10**, or to mark the pegs at **1/3/5** and cycle them round.

And we can even go into three dimensions. The natural three-dimensional analogue of ordinary solitaire, using a 135-cell "board" consisting of a 3 x 3 x 3 cube with a 3 x 3 x 2 knob on each face, is perhaps a game for theoretical analysis rather than practical play, though John Conway, Harry Davis, and I have all looked at it, Davis coining the name **Solidaire**, and I made myself a representational board in two dimensions back in the 1960s. This is described in *The Ins and Outs*, with some solutions to single-vacancy single-survivor problems. However, the game seems not to offer a great deal to justify its additional complication, expect that the finding of solutions in as few moves as possible might make an interesting challenge for computers, and some years ago David Singmaster sent me an advance publicity leaflet for a puzzle **Pyramad 6000** which appeared more practical. It was claimed as being "based on the ancient game of Solitaire, and the mysterious shape of Egypt's Pyramids", the idea being "to play a type of Solitaire, on each of the four sides in turn, ending up with one peg remaining on side four". In the absence of further details, we can only guess what it might have looked like, but the image conjured up is of a set of triangular lattice faces meeting at a point, ordinary triangular solitaire being played within and along the edges of each face but no peg being allowed to jump directly from one face into another.

If this was indeed the case, the selection of four sides was unfortunate. The natural problem to set on a pyramid is the apex problem, but this cannot be solved on a four-sided pyramid with triangular lattice sides however many levels we have; indeed, if we start with the initial vacancy at the apex, it is impossible to play down to a single survivor anywhere (proof by the usual *ABC* argument). Had the pyramid been given six sides instead of four, with some number of levels of the form $3n + 2$, the apex problem would have solvable by proceeding round the pyramid clearing a side at a time, and with five levels we would have had a challenging and attractive version of ordinary triangular solitaire on a 61-hole hexagonal board.

I don't know if the puzzle was ever put into production.

The future?

The history of peg solitaire is not yet complete, and new boards were being tried out even as this paper was being written. To any reader who is thinking of experimenting, I suggest that triangular lattice boards might be a particularly fruitful field; they offer plenty of scope, and only a few relatively obvious cases have yet been tried. And every now and then, I am sure that somebody will find himself trying to play solitaire on a network which he has come across for some quite different purpose, and will find that it yields attractive and enjoyably difficult problems.

Nor are the possibilities of square lattice boards exhausted. During the meeting, Tim Turner showed me two such boards which somebody had been to the trouble of making and of which I was previously unaware: a 69-hole board consisting of a 7 x 7 square with an additional 5 x 1 strip along each side, and a 65-hole board comprising a 5 x 5 square plus 5 x 2 extensions with alternate diagonals marked in the manner of an Alquerque board (a 5 x 5 square board on which you can move along the diagonals of length 5 or 3 but not along those of length 4 or 2). No doubt many others are out there somewhere. He also showed me a 55-hole board in the form of a right-angled triangle of side 10, but even though this was fully loaded with pegs there was nothing to indicate that its intended use was for playing solitaire, and on reflection I think it almost certainly wasn't.

Summary

We have highlighted recent discoveries in the history of "standard" peg solitaire, and we have looked at some of the many non-standard boards that have been used over the years. Had I known then what I know now, I would have written some of the historical sections of *The Ins and Outs* quite differently. I have therefore revised the paper in the light of feedback received, and am posting this revised version on the Peg Solitaire page of www.jsbeasley.co.uk so as to be available as a quarry to future writers on the game. Be it noted also that "Contributions towards a historical update", already posted on this site, contains further detail, including Internet links to some of the source material.

And I do wish that people would stop perpetuating the legend of the prisoner in the Bastille.

Appendix 1 (19 September 2014)

In this paper as originally posted, I wrote that I had not seen a relevant book by de Bouis, *Le Nouveau Jeu du Solitaire, réduit en problêmes géometriques et en décorations enluminées*, Paris, 1753. I have now seen this, and a brief account follows. Dic Sonneveld informed me that it had been digitized and made available on the Internet, and a Google search for "de bouis solitaire" produced a link through which it could be downloaded.

The book gives the rules of the game, the author believing them not to have been already published, followed by
- a set of 21 problems in which the player has to reduce the number of pegs on the board to 21, 20, 19, ..., 1 in turn, no further play being possible in any case;
- some further problems including one in which the penultimate move removes nine men;
- a suggestion for the enrichment of the game by the use of colours;
- a brief discussion of "marked peg" problems, in which the player nominates the peg which is to be the last survivor (he appears to have freedom in the choice of the hole to be initially vacated).

It contains neither information nor speculation about the game's origins.

The principal new material here appears to lie in the suggestions regarding the use of colours, which appear at least partially to anticipate the idea which John Maltby successfully patented over two hundred years later, and in the nomination of a marked peg to be the final survivor. Many of the problems were of course also new, but on the whole these do not strike me as being particularly interesting; the writer often contents himself with a final position having only lateral symmetry when a more attractive one with rectangular or even square symmetry could have been reached (and in some cases had been reached in the problems in the letters to *Mercure Galant*). The section on marked men starts with a note that the writer didn't believe it possible to nominate one of the pegs at **19**, **11**, **13**, **25**, or **27** to be the final survivor, but in fact all can be done though none is particularly easy.

The book appears to have had little influence, but it should certainly be included in any future bibliography of the game. The publisher was Nyon.

Appendix 2 (3 October 2014)

In respect of the 45-hole board, Fred Horn has sent me a picture of a board of this size and shape which his grandfather had made, and which is item 18028 in the HONGS collection (Historisch Overzicht Nederlandse GezelschapsSpellen). He thinks it was made sometime in the late 1920s or early 1930s, and he is sure that his grandfather would have had something that served as an example. However, this board appears to have been a private construction for use at home, not a surviving instance of something which had been on general sale, and it is not clear whether it provides evidence that that the 45-hole board was in vogue in the Netherlands at the time, thus refuting my scepticism regarding the article in *Algemeen Handelsblad*, or whether the article itself had provided the example which his grandfather had copied.

Appendix 3 (20 October 2014, corrected 8 January 2016)

After the meeting, James Dalgety drew my attention to item 03663 in the Slocum collection, which is a 19-hole circular board (outer ring of twelve holes, inner ring of six, central hole) with clay balls in the holes. Jerry had asked Len Gordon in 1985 whether it might have been a solitaire board, and James asked me the same question in respect of a similar item which had been brought to his attention. Well, nothing can be said for certain in the absence of any accompanying documentation, but it certainly plays very well as a solitaire board, and I have written it up elsewhere as **Clock Solitaire** with a selection of 24 varied problems. It makes a worthy addition to the fold, though it cannot be claimed as a historical item in the absence of further evidence.

Appendix 4 (27 May 2016)

A few more items have come my way.

George Bell recently drew my attention to an attractive pentagonal board which proved on investigation to have been developed by Merse Előd Gáspár and to have won second prize in a competition at the Fifth Hungarian Annual Puzzle Meeting at Bakonysárkány in 2011:

1

2 **3**

4 **5** **6** **7**

8

9 **10**

11 **12**

13

14 **15** **16**

Note that lines such as **2**-**5**-**8** are *not* carried across to the far corner, so that we can jump into and out of the centre but not across it. This board is far superior to the pentagram board which is Figure 14.13 in *The Ins and Outs*, and would certainly have been included in the paper had I been aware of it. The holes are more evenly spaced (a point of importance to board makers, however irrelevant it may be theoretically), and the problem "vacate the central hole and play to leave a single survivor there" can be solved. More generally, the holes divide into two sets, the inner six and the outer ten, and a single-vacancy single-survivor problem can be solved if and only if the initial vacancy and the hole to receive the final survivor are in the same set. Further problems can be found on the Peg Solitaire page of www.jsbeasley.co.uk. It is a very pleasant addition to the fold.

According to David Singmaster's *Sources in Recreational Mathematics*, a 1924 book *Fun, Mirth & Mystery* by R. A. Hummerston includes a peg solitaire game played on an octagram board. Examination shows this game to present some interesting features. I haven't seen the original source, and am relying on David's description. The game, called "Perplexity", uses a 16-hole board

1 **2**

3

4 **5** **6** **7**

8 **9**

10 **11** **12** **13**

14

15 **16**

with jumps allowed along rows, columns, and diagonals. The task given is to leave hole **16** vacant and play to leave a single survivor, and a further challenge is to specify where the final survivor is to be left. Even this is not particularly difficult, but the game has more to offer, and all single-vacancy single-survivor problems can be solved apart from "vacate and finish at **3**". I don't know if a proof of this one unsolvability has already been published, but I have posted one on the Peg Solitaire page of www.jsbeasley.co.uk.

Also attributed to Hummerston is a problem called "Leap frog" on a 4 x 3 board (four columns, three rows) with extra squares attached to the left-hand end of the top row and the right-hand end of the bottom row. The problem given is to put twelve white counters in the 4 x 3 area and a black counter in one of the extra cells, and to remove all but the black man. Even with the added statement that the problem can be solved in eight moves, this is pretty trivial, which raises the question: where should a survey like this draw the line?

The natural condition to impose is that somebody should have been to the trouble of actually making a board, but this would rule out Conway's splendid problem of the solitaire army (see page 7). More generally, almost all problems on square lattice boards can be explored using an ordinary chessboard, and other types of board can be drawn out on paper and counters or Scrabble tiles used for the men. So physical realisation in wood or some other durable material cannot be a necessary condition for inclusion, and all one can do is to select games which the compiler finds interesting for one reason or another. If this means that different people's selections vary, so be it.

David Singmaster's *Sources* also give references to a prior publication of Star Jump (page 10) as "Checker Star" in a 1978 book *Puzzle Fun* edited by G. R. Putnam, and to a wooden 37-hole board in Edward Hordern's collection on the back of which is inscribed "Invented by Lord Derwentwater when Imprisoned in the Tower". But the "Lord Derwentwater" in question would appear to have been the third Earl, who joined the 1715 rising, and was captured at Preston, imprisoned in the Tower, and executed in February 1716. Since the game was fashionable at the French Court eighteen years earlier, this is clearly worthless as a statement about its origins, though it does show how easily picturesque legends about the invention of games can arise and become current.

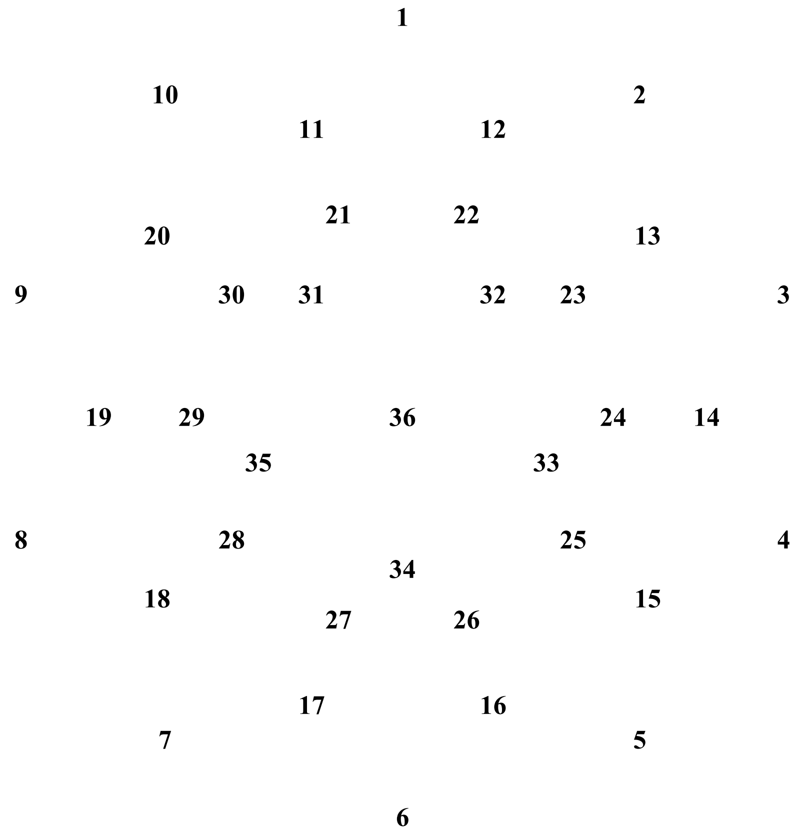
Appendix 5 (17 April 2017)

In a recent paper "Clock Solitaire" in issue 101 (March 2017) of *Cubism For Fun*, George Bell questions whether Clock Solitaire in its original form (Appendix 3) was truly intended as a jumping puzzle, on the grounds that the holes are too close to allow the easy picking up of individual marbles. On the evidence of the photograph he reproduces, the distance between adjacent marbles is barely a fifth of their diameter, or only about an eighth of an inch given the board size as reported, and this would seem to bear out his reservations.

George also notes that the thirteen-hole diamond board with diagonal moves shown on page 10 (and discussed in chapter 10 of *The Ins and Outs*) was patented by William Breitenbach as far back as 1899, and marketed as "The Great 13 Puzzle".

Appendix 6 (17 April 2017)

James Dalgety has drawn my attention to an **Ancient Star Problem Puzzle** in the Hordern-Dalgety collection. It was marketed in 1949 by Reflex Products Company of Cleveland, Ohio, and uses a 36-hole board based on three concentric rings as shown below:



Holes **1-10** form an outer ring, holes **11-20** an intermediate ring, holes **21-30** an inner ring, and holes **31-35** a pentagon. There are five outer-ring-to-outer-ring lines across the centre typified by **1-36-34-6**, five intermediate-ring-to-intermediate-ring lines typified by **19-29-36-24-14**, a pentagram **1-5-9-3-7-1** each of whose lines contains six holes (typically **9-30-31-32-23-3**), and a pentagram **2-6-10-4-8-2** each of whse lines contains four holes only (typically **8-28-25-4**). All this gives a total of 170 possible jumps (twenty around each of the three rings, twenty along the four-hole lines through the centre, thirty along the five-hile lines through the centre, and forty and twenty along the lines of the pentagrams).

The board was supplied with 28 white pegs and two blue, and with a small selection of problems in which the task is to reduce to a single peg in the centre. If a problem includes one blue peg, this is to be the final survivor; if it includes both blue pegs, one is to capture the other as the last jump. Prizes were offered for further problems sent in by players.

There are some useful block moves and removals. A set of pegs in the outer ring **1-10** plus the pentagon **31-35** can be reduced to a single peg in **36**, (play for example **2-36, 7-32**, **4-36**, **32-7**, **6-36**, **7-32**, **8-36**, **32-7**, **10-2-4-6-8-10-36**). If hole **1** is empty and the rest of the outer ring is full, this ring can be reduced by jumps along it to a single peg in hole **2**, **4**, **5**, **7**, **8**, or **10**, but not to a peg in hole **1** itself nor to a peg in hole **3**, **6**, or **9**, and an equivalent is true of the intermediate and inner rings. If the central hole and one pentagon hole have different contents, one being full and one empty, the outer ring pegs on the same line can be removed; if any two of the holes **31-36** have different contents, the entire outer ring can be removed.

With the aid of these and of the normal line-of-three removal, the centre-peg problem is not difficult, and I suspect that all single-vacancy single-survivor problems are solvable thought I have not verified every case. There are attractive problems where the centre is vacated and the objective is to leave some symmetrical or other interesting pattern; for example, we can play just to leave the outer ring **1-10**, or just the intermediate ring **11-20**, or just the inner ring **21-30**, or just the pentagon **31-35**. There are also "swap" problems; for example, we can vacate **36**, put blue pegs in **1** and **6** and white pegs everywhere else, and play to interchange the blue pegs and clear the rest of the board.

Why the manufacturer did not supply a full set of white pegs is not obvious.