Clock Solitaire

by George Bell

> "*Anyone can make a hard puzzle with lots of complicated pieces but how can you possibly make a hard puzzle out of a few easy pieces? When a seemingly simple puzzle is unexpectedly difficult, it's usually because, as well as the obvious problem, there are some hidden ones to be attended to*." [1, p. 844]

This puzzle mystery begins with an object in the Jerry Slocum Puzzle Collection. Slocum purchased a used copy in 1980 in London [2]; another copy of this whatsit is shown in Figure 1. It is a brass board 5" in diameter with 19 depressions or "holes", together with a colourful set of clay marbles (not found with the original board). These boards are estimated to have been manufactured around 100 years ago. No instructions or other documentation were found with either copy.

Figure 1. The mystery object (photo courtesy St. John Stimson).

As metagrobologists we are quick to interpret this as a board game or puzzle. However, the original use of this object may well have been entirely different. As a peg solitaire board, the holes seem a bit close together for easy marble grabbing, and there are no lines indicating jumping rules. James Dalgety suggests that it could have been a "WW1 jig for selecting steel balls from a bulk container ready for assembly into a ball race", see puzzlemuseum [3].

Clock Solitaire rules

If we interpret the mystery object as a peg solitaire board, what are its jumping rules? This is not obvious, given the round nature of the board. In 1985, Jerry Slocum asked peg solitaire expert Leonard Gordon this question.

Before we consider Gordon's response, let us briefly review the rules and terminology of peg solitaire: The holes are filled with pegs (or marbles), leaving one empty, called the **starting vacancy**. The player then jumps a peg over an adjacent peg into an empty hole, removing the peg that was jumped over. We call one or more consecutive jumps by the same peg a **move**. The goal is to choose a sequence of jumps which finish with one peg; if this achieved, the hole where the last peg ends at is called the **finishing hole**. A **complement** problem begins with one peg missing and finishes with one peg at the starting vacancy; the **central game** is the most symmetric complement problem beginning and ending in the centre.

Leonard Gordon's response is preserved in the Slocum collection [2]: he notes that the 19-hole board can be interpreted as a hexagonal board with 3 holes on a side, with additional jumps added around the rim. This gives a set of 66 possible jumps (counting every conceivable jump, even if pegs aren't in the proper configuration), and Gordon's letter gives a solution to the central game.

One might assume this settled the matter. However, in 2014, another peg solitaire expert, John Beasley, re-examined this board. He was unsatisfied with the 66 jump set, because he felt it made the puzzle too easy(!). He removed all of the interior jumps except for six along each of three radial lines. This forms a set of 42 jumps. Because of the resemblance to a clock, he called it "*Clock Solitaire*". The hole numbering was taken from a clock face, with the inner circle showing 24 hour times.

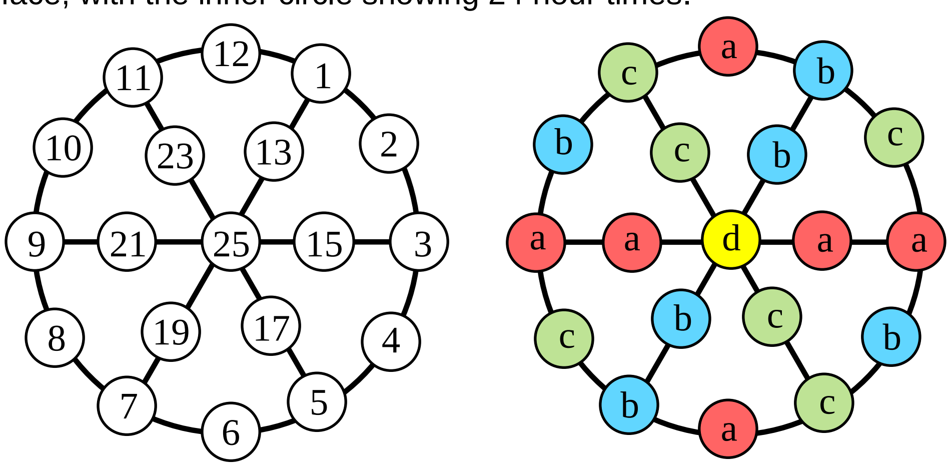

Figure 2. *Clock Solitaire* hole numbering, and colouring for analysis.

Our diagrams show straight or curved lines along which legal jumps may be made (jumping "around a corner" is not allowed). We note that the 66-jump rule set adds 24 jumps along the "lines" 10-23-13-2 and similar. In what follows I will try to convince you that the 42-jump rule set makes a superior puzzle. Of course, Douglas Adams fans know 42 is the answer to **The Ultimate Question** [4]!

Clock Solitaire analysed

Given a board position, let N_a be the number of pegs in the red holes (a), and similarly for N_b, N_c and N_d. Note that N_d indicates whether the centre is occupied (and we will ignore it for the moment). What happens to N_a, N_b and N_c after a solitaire jump is executed? If the jump occurs along the outer ring (holes 1-12) then two of these N's decrease by 1 while the third increases by 1. If we add the N's in pairs, we see that the parity (even or oddness) of the three sums: ($N_b + N_c$, $N_a + N_c$, $N_a + N_b$) *does not change as the game is played*. For an interior jump, one of the N's may decrease by 2 or stay the same, while the others stay the same. So after any jump is executed, the parity of all three sums does not change.

Suppose that the starting vacancy is red (a). Then N_a=5 and N_b=N_c=6, therefore the starting parity is (even, odd, odd) or (0,1,1). The only one peg states with this parity have one peg in a red hole (a). We have proved the

> **<u>Colour finish rule</u>**: If the starting vacancy is red (a), we can only finish at a red hole (a).

This also holds for blue and green (b and c), which must be the case since the labelling is rotationally symmetric, but what about yellow (d)? If we begin with only the centre peg missing, the starting parity is (0,0,0), and the only one peg state with this parity is one peg in the centre. Therefore, the colour finish rule also holds for the yellow hole (d), namely the centre.

We have now determined which problems are potentially solvable, and it turns out that all problems beginning and ending at the same colour are solvable. You can show this by finding solutions to all such problems. For example, starting with the north pole 12 vacant, you can finish at 12, 9, 21, or 6 (or the mirror symmetric holes 15 and 3).

Now suppose instead we use the 66-jump set proposed by Leonard Gordon. The parity arguments above are no longer valid, and it turns out that we can begin from any starting vacancy, and finish at any hole. I have demonstrated this using a computer program which explores all possible jump sequences. While this may seem an interesting property for the puzzle, it is not a good sign.

Something similar happens in regular peg solitaire. Here the possible finishing holes are dictated by the "rule of three" [1]. If one allows diagonal jumps, the rule of three is no longer valid—it is possible to begin from any vacancy, and finish at any hole [5]. Diagonal jumps make the puzzle too complicated, although this is a matter of opinion.

In *Clock Solitaire*, by reducing the jump set from 66 to 42, John Beasley simplified the puzzle. How can simplifying a puzzle improve it? The reason is given most elegantly in the quote given at the start of this document. The Finishing Rule is a "hidden problem" which must be attended to.

John Beasley has come up with many nice problems for *Clock Solitaire* [6]. The problems include classic ones starting with one peg missing and finishing with one peg. There are also problems beginning with the centre vacant and finishing with pegs in a symmetrical pattern. In *Clock Solitaire* it is possible to finish in some symmetrical pattern with the centre vacant, and also finish with the same pattern with the centre filled. This is unusual, and would be impossible in standard peg solitaire. Also included are “man on the watch” problems, where the last peg is required to capture all remaining pegs in the final move.

The big picture

Many people assume it is easy to start with an empty board, add holes and lines in some nice pattern to create a great new peg solitaire puzzle. This is not the case! It is actually rather unlikely you will end up with a good puzzle.

Here are some **desirable requirements** for a peg solitaire board:

1. The board must have square or hexagonal symmetry.
2. There must be a hole at the geometrical centre of the board.
3. It must be a reasonable size. Less than 35 holes seems reasonable to me.
4. The central game must be solvable. Curiously, this is the only requirement which references the rules of peg solitaire.

Surprisingly, there are not many boards satisfying these four requirements. There are two cases to consider:

Case 1) The holes and jumps lie on a regular grid (square or triangular).
Case 2) Anything goes (holes and jumps arbitrary, but still symmetric, of course).

Suppose we consider Case 1, and also require that the board have no internal voids or “missing holes” (a technical, but important detail). The term I use for such a board is **gapless**. It has been shown [7] that there is only one gapless board which satisfies the requirements: the standard 33-hole cross-shaped board (aka the “English board”).

Almost by definition, Case 2 boards have unevenly spaced holes, so some jumps will be longer than others. We define the **jump ratio** of a board as the length of the longest jump divided by the length of the shortest jump. I prefer boards with a jump ratio closer to 1, but this is an aesthetic matter. *Clock Solitaire* falls under Case 2 and has a jump ratio of π/3 ~ 1.05. We now present several other Case 2 boards which satisfy the four desirable requirements.

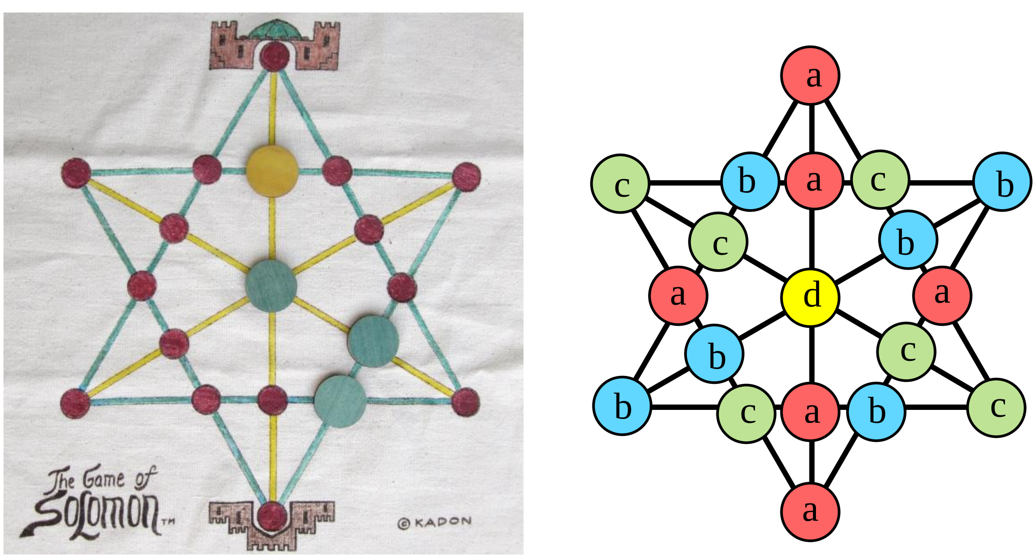

Figure 3. A fabric Solomon board (from Kadon [8]), colouring for analysis

Solomon is a board game invented around 1970 by Martin Gardner. The board [8] is based on a 6-pointed star or hexagram (Figure 3). The 19 holes are far from evenly spaced, and *Solomon* is unusual in other ways. In standard peg solitaire, if peg 1 can jump peg 2, then it is impossible for peg 1 to occupy the hole vacated by peg 2, even if it performs any number of jumps. With *Solomon*, this is possible. Figure 3a shows such a situation, here the top (yellow) peg can capture the centre peg, and in two more jumps it can occupy the centre. We call this odd move an **ouroboros** move, after the mythical snake eating its own tail.

There is an analogous a-d labelling of Solomon (Figure 3b) which obeys the colour finishing rule. It turns out that all problems beginning and ending at the same colour are solvable.

Round Solitaire [9] was invented around 2009 by Tetsuro Kawahara (Figure 4a). This board has nearly equal jump lengths and admits ouroboros moves. *Round Solitaire* can be drawn with the outer ring of 12 holes on a circle [10], which reduces its jump ratio slightly to 1.21. The usual parity argument on the 12 red (x) holes shows that they obey the colour finish rule [11], as does their complement. As with *Clock Solitaire* and *Solomon*, all such problems are solvable.

The *Crystal Palace Wheel Puzzle* [12] (Figure 4b) was purchased by Jerry Slocum in 1993 from Ray Bathke. John Beasley invented this 13-hole board, as well as a nice set of problems for it. Beasley allows jumps along the outer ring as well as the inner ring (total: 42 jumps), one can optionally remove those along the inner ring (total: 30 jumps). Jumps along the outer ring are over twice as long as those along a diameter. The central game is solvable in 5 moves with either jump set, and the usual parity arguments show which problems are solvable.

Figure 4. Round Solitaire (2009), The Crystal Palace Wheel Puzzle (1990), Hoppers (1999). The first two photos are courtesy the Slocum Collection.

One final board is *Hoppers*, made by ThinkFun [13]. This board was patented in 1899 by William Breitenbach and marketed as *The Great 13 Puzzle*. A century later, Nob Yoshigahara rediscovered this board and added the red frog, which must be the final survivor. Hoppers was then re-issued in 2013 with 40 all-new challenges using only green frogs. Solvable problems are given in [5] and [11], with playing tips in [14].

The ultimate challenge on these boards is to solve the central game in as few moves as possible. The length of the shortest solution has been found by my computer solver and is given in Table 1 under "Min CG". The column heading "Central Finishes" is the number of finishing holes starting with the centre vacant (symmetrical locations are not counted as different). “All Comp?” indicates whether all complement problems are solvable.

Board Name	Holes	Jumps	Jump Ratio	Ouroboros Moves?	Min CG	Central Finishes	All Comp?
Standard	33	76	1.00	No	18	2	Yes
Round	21	60	1.21	Yes	8	3	Yes
Solomon	19	54	1.73	Yes	8	1	Yes
Clock	19	42	1.05	No	8	1	Yes
Breitenbach	17	40	2.00	Yes	6	1	No
Hoppers	13	32	1.41	No	7	2	No
Wheel	13	30 or 42	2.09	Yes	5	1	Yes

Table 1. A comparison of boards which meet the desirable requirements.

I should point out there are a few more Case 2 boards which meet the desirable requirements. A 25-hole board appears at the end of [7] which is not gapless, it could be considered Case 2. One can take the standard 33-hole board and add additional jumps, not along grid lines, to obtain a Case 2 board which still satisfies the four requirements. We have not included these extensions, one of which is called *Hyper Solitaire* [10], and another is standard 33-hole peg solitaire with diagonal jumps allowed.

After this article was published, Michael Desilets pointed me to Breitenbach’s Board (Figure 5). This 17-hole board was patented in in 1899 by W. C. Breitenbach [15] but has since fallen into obscurity. Figure 5 (right) shows a labelling of Breitenbach’s Board which obeys the colour finishing rule. Detailed analysis shows that all problems beginning and ending at the same colour are solvable, with the exception of those of the form: begin at 4 and end at 4 or 14. A fun challenge is to find a solution to the central game in 6 moves.

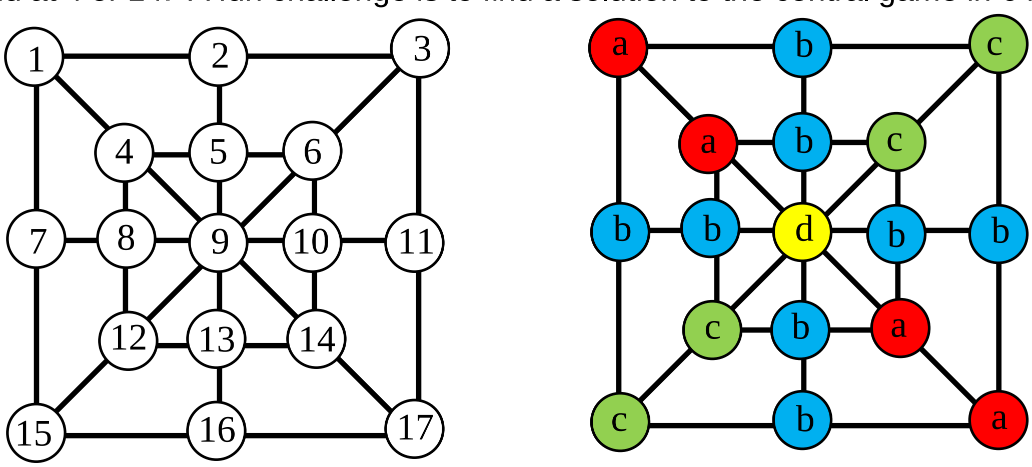

Figure 5. Breitenbach’s Board (1899), hole numbering and colouring for analysis

Summary

Clock Solitaire began as an obscure brass board, only a few copies of which have been found. Possibly it wasn’t even intended as a puzzle. A century later, John Beasley distilled down a nice set of jumps for this board, which results in an elegant peg solitaire puzzle. Good peg solitaire boards are rare, and in my opinion *Clock Solitaire* takes its place among the best.

References

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