Solving Puzzles Using Maps

by George Bell

When driving in an unfamiliar city, one consults a highway map. When faced with an unfamiliar puzzle, wouldn't it be nice to have a map of it? I claim that when we solve a disassembly puzzle, we instinctively create a mental map of the puzzle. I think of this mental map as a graph, with the nodes being the possible states of the puzzle, and edges representing physical motions of the puzzle pieces which change the puzzle from one state to another. After we solve a puzzle and have this mental map to follow, solving it a second time tends to be significantly easier.

Why not write down this mental map to help other people navigate the puzzle? The problem is that many puzzles have too many possible states, or there may be no simple way to describe a particular state of the puzzle. For example, the physical states of a burr puzzle tend to be difficult to describe. The solution is usually given as a sequence of motions of the puzzle pieces, often communicated by drawings. Such a solution gives the sequence of edges followed in a path to the desired goal state (namely, disassembly). This is not a map of the puzzle, but analogous to turn-by-turn directions to take you from point A to point B, and turn-by-turn directions work perfectly well—provided you don't make any wrong turns.

Is it ever possible to write down a map of a puzzle? One class of puzzle where mapping does work is sliding block puzzles. Here a subset of possible puzzle states is generally selected, and the edges connecting them represent multiple sliding moves of the pieces. A map of the Century Puzzle covers three pages [1]. This map is superior to a list of 100 moves needed to solve the puzzle. For one thing it describes all possible states that can be reached by the puzzle. So even if you are not in the starting state, you should be able to navigate to the solution.

Another class of puzzle where maps work well are disentanglement puzzles involving rings or disks. The number of possible states for such a puzzle is often quite small and the states easy to identify. The difficulty lies mainly in figuring out what moves are possible, and whether they take you closer or farther from the goal. I will now describe two such puzzles that can be solved by creating a solution map.

Hanayama's Cast Duet

This fascinating puzzle was designed by Oskar van Deventer. The puzzle consists of a metal grid with 9 holes (see Figure 1), and what at first appears to be a single ring. The ring is actually split like a bagel into identical halves which are magnetically connected. The rings also contain a radial gap so that they can be inserted onto the grid. The rings have a point which sticks into this gap (see Figure 2), allowing them to move along various slots between holes in the grid in one orientation only.

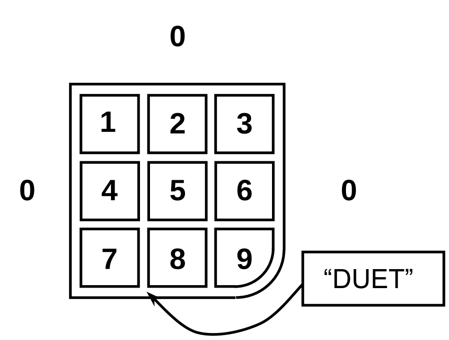

Figure 1. Cast Duet grid

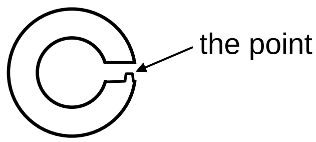

Figure 2. Cast Duet rings

We number the holes in the metal grid as shown in Figure 1, where the outside of the grid is “hole 0”. The grid itself is always kept in the orientation in Figure 1 (it may be tempting to turn it over to look at the slots on the bottom, but this is not necessary once you can read the map). The position of a ring in the puzzle is denoted by a two digit number describing which two holes the ring spans. The order of the two numbers is chosen so that, when the ring is spun around so that the point is up (toward you in Figure 1), the second number marks the location of the point. For example, when the ring is off the grid, we can think of it as in state “00”. We can place the ring on the grid by sliding it with point up onto hole 1, to state “01”, or with point down onto hole 8, to state “80”.

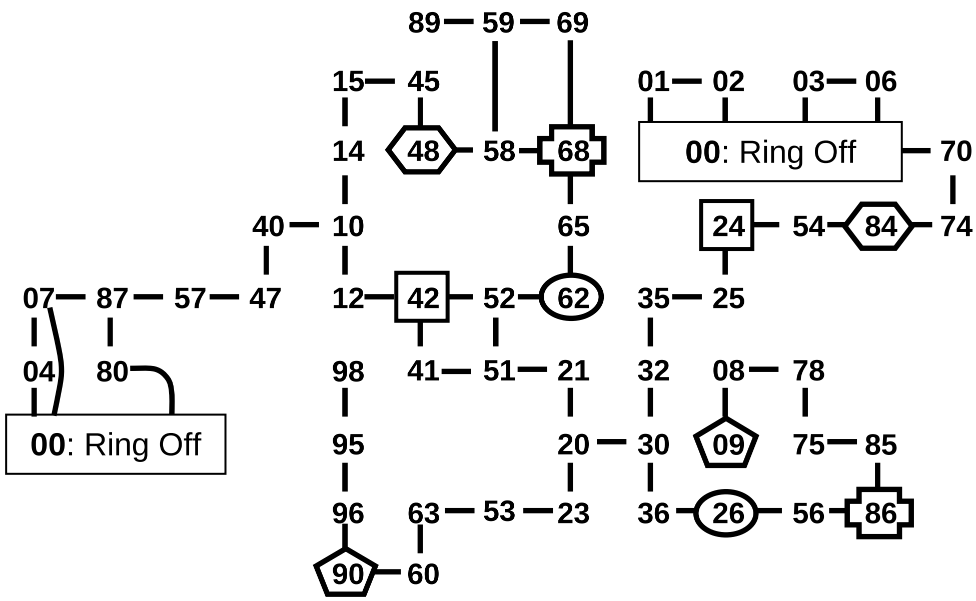

Figure 3. A map of the Cast Duet showing all positions of a single ring

Figure 3 is a map of all the positions that can be reached by a single ring. Horizontal moves in the map are done with the point down, and vertical moves with the point up. Thus, any transition from a horizontal to vertical move (or vice-versa) requires a 180° rotation of the ring.

This is a very simple graph with few 3-way intersections. Solving the puzzle requires placement (or removal) of **both** rings, which is slightly more complicated. If you wish to place the connected ring between holes *x* and *y*, you need to navigate one ring to *xy* and the other to *yx*. There are two problems which may occur, first the presence of the first ring may impede the other. Second, the two rings may end up “back-to-back”, unable to interlock. The first problem is usually easy to avoid, for the second problem, try placing the rings in the opposite order.

As sold, the puzzle has 5 problems in increasing level of difficulty for paired-ring placement (the most difficult being the starting position of the puzzle, 90/09). These 5 problems are denoted in Figure 3 by paired node symbols. In fact, there are 28 possible ring placements, 8 off the edge of the grid, plus 20 internal locations: 6 horizontal, 6 vertical, and 8 diagonal. The graph in Figure 3 has 56 nodes (and 00), indicating that we can place a single ring in either of two orientations in all 28 locations. The grid seems to be designed with this in mind, because not all slots in the grid are needed to solve the 5 problems that come with the puzzle.

In fact, all 28 paired-ring positions can be reached, as I have verified by solving each in turn. Most cases are not difficult with the map, the most difficult being 14/41, and 98/89. It is quite difficult to reach these positions because the two rings block one another. I recommend these two difficult problems as they require solution techniques beyond the original 5 problems[1].

Hanayama's Cast Disk

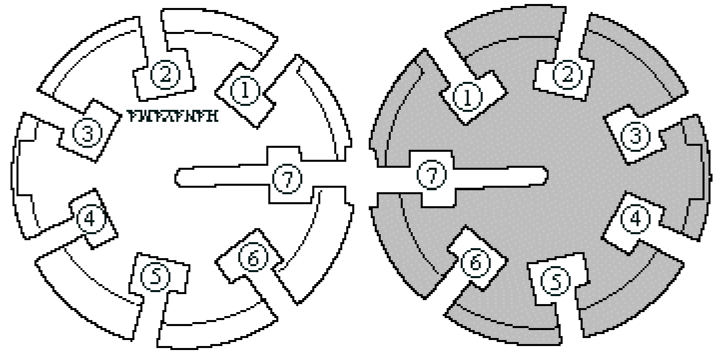

Figure 4. Notch labels for the two disks

Another invention of Oskar van Deventer, the Cast Disk puzzle consists of two metal disks, each with seven notches in the edge. Only in two special notches (labelled "1") can the disks be separated, normally it is possible to rotate the disks between notches but not separate them. Two other special notches (labelled "7") are extra-long and allow the two disks to slide together into the spherical, interlocked, starting position.

The two disks are very similar but are not identical, they can be differentiated by careful comparison of the notch labelled "4". The left disk is usually labelled "®HANAYAMA", but unfortunately not all copies of the puzzle have this label. For what follows this disk is always to be held in the left hand. The notch-labelling scheme in Figure 4 is used in the Hanayama solution sheet (from Puzzlemaster [2]), which calls the Hanayama disk "B" and the other disk "A". To connect the disks, tilt the top of the right disk in Figure 4 toward you, and the bottom away from you.

Figure 5 shows a map of all 49 positions that can be reached by this puzzle (every pair of notches can be connected). The 2 digit notation *xy* means that notch *x* in the "Hanayama" disk is connected to notch *y* in the other disk. The b**L**ack links denote rotations of the **L**eft disk, g**R**ey links rotations of the **R**ight disk (if grey cannot be distinguished from black in this print of the article, the disk to rotate can also be identified by which digit changes). I find it interesting that a puzzle which is physically two interlocked rings has a map that is four interlocked rings! This is a really a fairly simple maze, with few 3-way intersections, and most people wander somewhat blindly about it, finding a solution without too much trouble (my son solved it this way at age 6).

One problem using this map is that, looking at the puzzle, it is not easy to tell which two notches are interconnected, and consequently where you are in the map. This can be solved by sticking tiny labels at each notch (which I did to create the map), but I will now show you how to solve the puzzle without resorting to such aids.

[1] **Solution for 98/89**: move a ring to 98, then the second to 65 (easy), then to 68 and 69. Moving from 68 to 69 is difficult because it seems blocked by the other ring, but it is possible. Then move the second ring easily to 59 and 89. 14/41 can be solved using a similar technique.

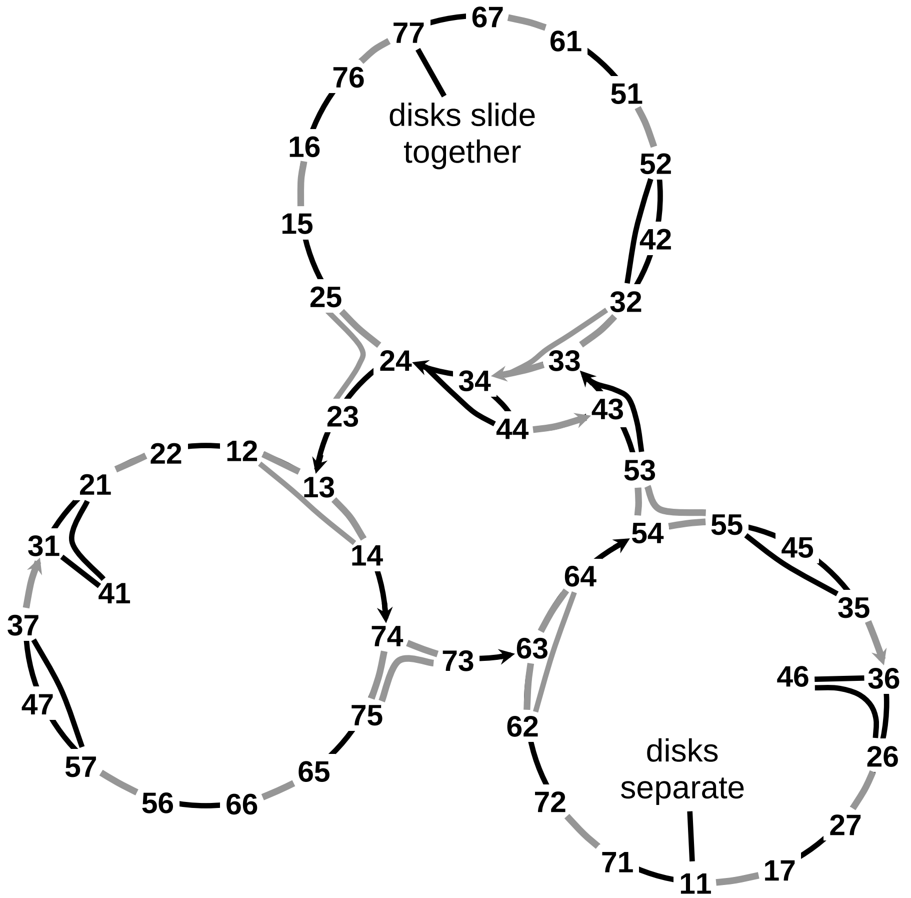

Figure 5. A map of the Cast Disk showing which notches are interlocked.

Holding the puzzle as you always should, with the Hanayama disk in your left hand, suppose I give you the instructions "$\mathbf{L^2RL^2R}$", abbreviated "$\mathbf{(L^2R)^2}$". This means, "turn the left disk two notches" followed by "turn the right disk one notch", then repeat. "But wait!", you say, "which **direction** do I turn each disk?" While this can be a problem in general, I can choose a **special** set of moves, where at any point, there is only one way to turn the next disk. I call this an **unambiguous** move sequence, because you can never be confused as to how to interpret the next move.

The parts of the graph where confusion occurs are marked by arrows. If I enter a node along such an arrow by turning one disk, I can then turn the opposite disk in **either** direction. Thus, in order to know which way to proceed, I will need to figure out which way to turn the next disk. Unambiguous move sequences are exactly those that avoid these arrows (moving against an arrow is perfectly valid and is not ambiguous). One unambiguous disassembly sequence is $\mathbf{(LR)^2(L^2R)^2LR^2LRL}$. You should be able to follow this sequence through the map to verify that it never follows an arrow into a node.

Note that any unambiguous sequence simply alternates left and right turns, moving usually one but occasionally two notches. The above sequence can be more easily remembered by counting moves: "**start with the left, then alternate single moves, except for moves 5, 7, and 10, which are double moves**".

If we try to reverse this sequence to reassemble the puzzle, we discover that the assembly is ambiguous. In fact, by looking at the map we see that there is **no** move sequence that disassembles the puzzle which is unambiguous both forward and backward. This may seem a sad state of affairs, until we realize that there is no reason why we **have** to assemble the puzzle using the reverse of the disassembly. You can assemble the puzzle using the unambiguous, simpler sequence: **$(LR)^4LR^2(LR)^2$**.

If you have a friend who thinks this puzzle is easy, one devious trick is to flip one of the disks 180° in the unassembled state (**warning**: this is also easy to do inadvertently). The puzzle can then be assembled to the interlock position 77, and while the map is topologically similar to that in Figure 5, the details are completely different, and someone who has memorized the above sequences will become lost. However, there is a good chance your friend may not even notice the difference if he solves the puzzle with the usual "maze-wandering" technique! In any event, unambiguous assembly and disassembly sequences can be found in Table 1.

Puzzle Orientation	Solution Type	Unambiguous Sequence	Start with	Double move #'s	Number of moves*
normal	assembly	**$(LR)^4LR^2(LR)^2$**	L	10	14
normal	disassembly	**$(LR)^2(L^2R)^2LR^2LRL$**	L	5, 7, 10	13
flipped	assembly	**$(RL)^4RL^2R^2(LR)^2$**	R	10, 11	15
flipped	disassembly	**$(RL)^4R^2LR$**	R	9	11

(*) Two consecutive moves by the same disk are counted as a single move.

Table 1. Summary of unambiguous move sequences for the Cast Disk.

It is critical to be able to recognise the "flipped state" from the starting (77) or finishing positions (11). If you are in the flipped state, and make one move from 77, you will find yourself at 17 or 71 (instead of 67 or 76 as in Figure 5). Similarly for the disassembled, flipped state, 11 leads to 12 or 21.

If you have multiple copies of this puzzle, you can interlock more than two disks! Each pair of disks will interlock using the map of Figure 5 (or the flipped map), but certain moves may be blocked by the other disk(s). If your friend was not impressed by your first attempt, borrow his puzzle and return it fully assembled, but with a second (or third!) fully assembled puzzle linked to it. This can be accomplished trivially by taking two assembled puzzles, and interlocking their 1 notches in the flipped state and moving to 12 or 21. It can also be done in more subtle ways that are much harder to disassemble. If you understand how to read the solution map in Figure 5, you should be able to figure out how to do this.

More Mapping

Solution maps are a useful tool for solving as well as designing certain types of puzzles. Many puzzles involving rings and plates, or route-finding can potentially be mapped. Often, it can be difficult to come up with a notation for the positions that is easy to interpret. I was inspired to map the above puzzles by Rob Stegmann's map of the Hanayama Cast Plate. You can view it and several other puzzle maps on his web site [3].

References

[1] E. Berlekamp, J. Conway and R. Guy, *Winning Ways for Your Mathematical Plays,* 2nd ed., Vol. 4, A K Peters, Wellesley, MA, 2004, pp. 882-884.

[2] Puzzle Master Inc., http://www.puzzlemaster.ca/

[3] Rob Stegmann's Puzzle Page, http://home.comcast.net/~stegmann/routefind.htm

Postscript

In 2010 (after this article was published) I received an email from David Peck with his own map of the Cast Disk. He made his map on a torus, and it looks quite different from my map; but they are, of course, topologically the same. Here is David's map converted to my notation:

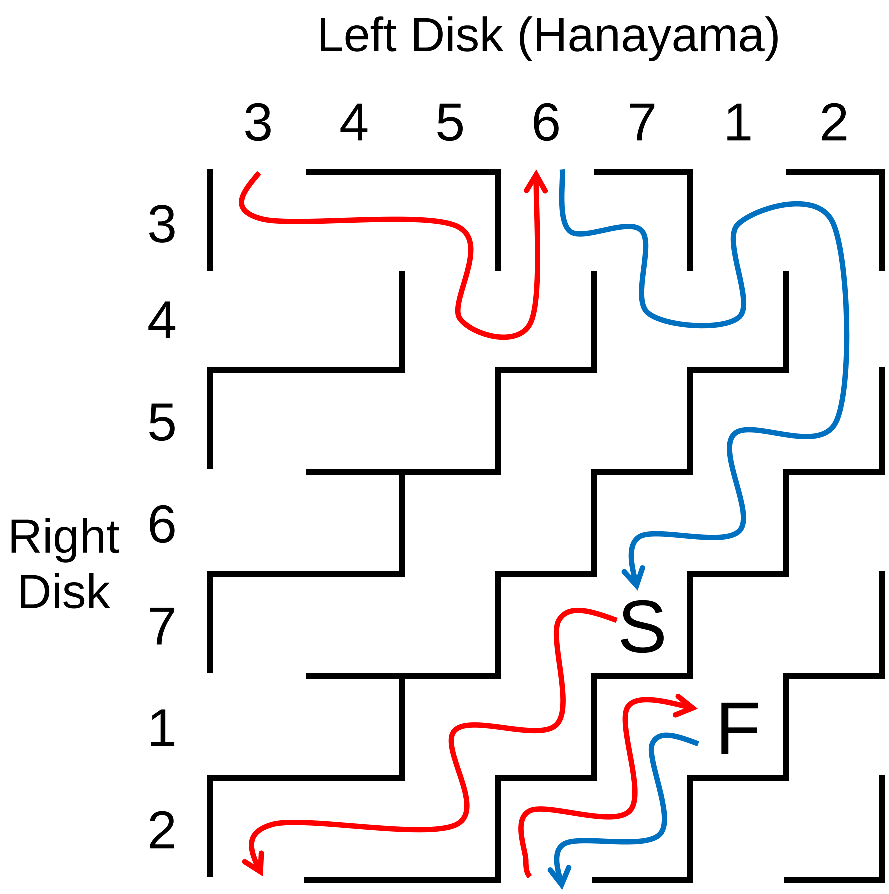

Figure 5. David Peck's map of the Cast Disk, drawn on a torus.

In Figure 5, "S" denotes the starting position where the disks are interlocked, and "F" the final position where the disks are separated. Since the map is on a torus, you can move off the top (or right) edge to reappear on the bottom (or left). The red path shows the unambiguous disassembly sequence in Table 1, and the blue path traces the unambiguous assembly sequence. Note that an arrow in Figure 4 corresponds in Figure 5 to entering a T-junction from the bottom.

Finally, Figure 6 shows my map for the Cast Disk when one of the disks is flipped.

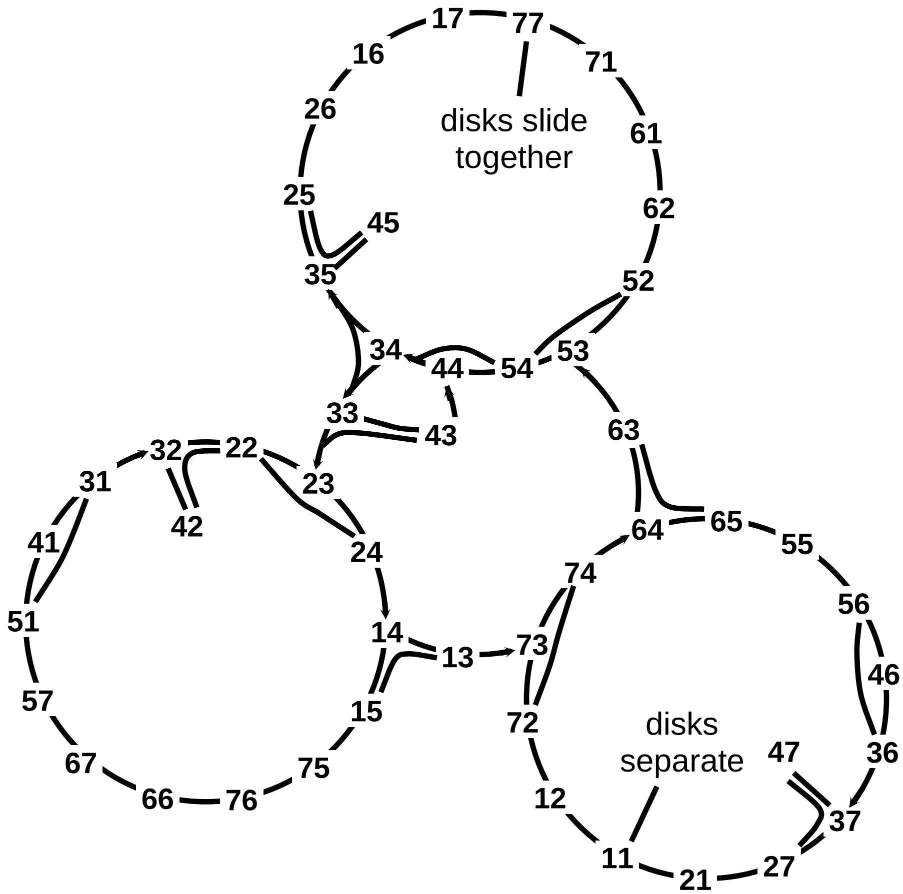

Figure 6. My map for the Cast Disk puzzle when the disks are flipped.