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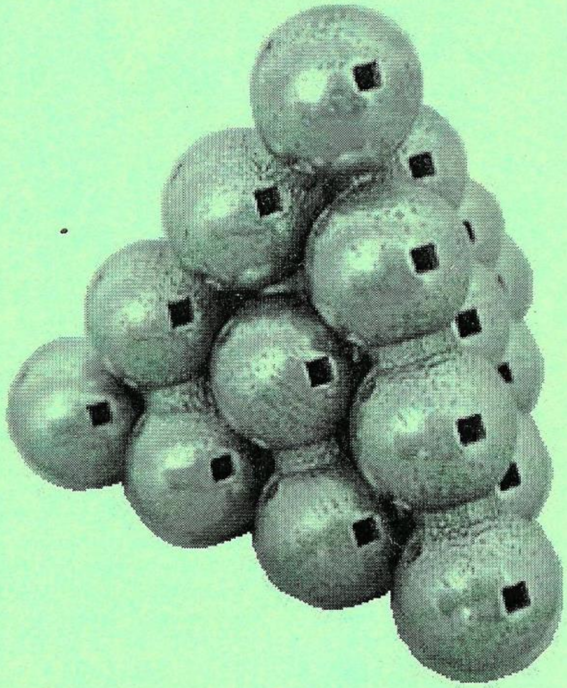
Cubism For Fun



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Classification of Polyspheres

by George Bell

Polyspheres are polyforms made by connecting spheres in the face-centered cubic (FCC) lattice (often called “cubic close packing”). Polyspheres are interesting objects from which to create 3D puzzles. As we will see, polyspheres include, in a sense, all polyominoes and all polyhexes. We can slice the FCC lattice along certain planes to obtain tetrahedrons, octahedrons or cubes. Thus, in polyspheres, we have a set of puzzle pieces that can potentially build three of the Platonic solids (to be precise, we obtain analogs of these solids made from stacked spheres). Many other interesting shapes such as 4-sided pyramids (or half-octahedrons) and cuboctahedrons are also possible. It is somewhat surprising that CFF is the only publication in which polysphere articles have appeared [1-5].

A convenient way to think of the FCC lattice is as edge-joined cubes (Figure 1). We consider all unit cubes with centers at integer coordinates (x, y, z) where the sum of the coordinates $x + y + z$ is even. In its “Spheres” geometry, BurrTools [6] uses exactly such an internal representation. The restriction on the sum of the coordinates eliminates half the cube locations, and ensures that cubes can only be joined along an edge. Of course, when displaying in the “Spheres” geometry, BurrTools replaces each cube by a sphere of diameter $\sqrt{2}$.

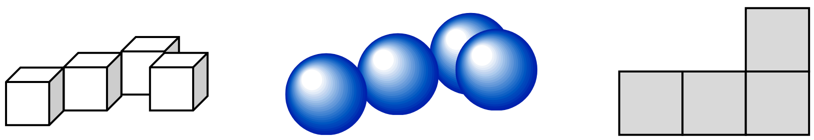

Figure 1. Four edge-joined cubes (left), the generated tetrasphere (middle) and the related tetromino (right).

But this is only the representation of the pieces. In BurrTools, the way pieces are handled (comparison, rotation, etc.) differs a lot from the “cube” geometry. Figure 2 (left) shows two configurations of three edge-joined cubes. These two objects are not identical—they are mirror images of one another. A mathematician would say they are **chiral** (a term for any 3D object not identical to its mirror image). When spheres are substituted for cubes (middle) the resulting polyspheres are identical (this solid is not chiral, or **achiral**). Although these puzzle pieces are 3D objects, the three cube or sphere centers lie in a plane (a **hexagonal plane** in which the spheres are packed hexagonally), so all of these pieces are related to the same trihex (right).

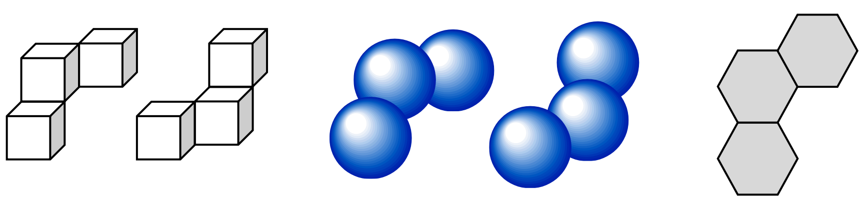

Figure 2. Three edge-joined cubes (left), the generated trispheres (middle) and the related trihex (right).

Two spheres touch at a point, so to make physical polysphere puzzle pieces one usually adds a cylinder to make a strong connection. This is now done automatically by BurrTools [6] in its STL export; most woodworkers use wood dowels. We note that a cylinder of any diameter can be used up to the sphere radius.

If we imagine spheres packed in the FCC lattice, and allow them to expand at the same rate, filling all remaining gaps, each sphere becomes a rhombic dodecahedron (RD). This 12-sided space-filling solid packs in the geometry of the FCC lattice, and can also be obtained by starting from a cube and beveling the 12 edges. We can also join rhombic dodecahedrons to make puzzle pieces [7], we call them "**polyRD**".

Polyspheres and PolyRD are closely related, but are not identical. If we take any polysphere, we can convert it to a PolyRD by building it out of RD's (because both share the FCC lattice). The subtle issue is that there can be more than one way to do this. For example, when the Figure 2 polysphere is converted to polyRD it becomes chiral, with two mirror-image versions possible. A remarkable fact found by Torsten Sillke [8] is that the **only** polyspheres which become chiral when converted to polyRD *lie in the hexagonal plane* (so they are all related to polyhexes). All other polyspheres correspond to a *unique* polyRD.

We should point out that these conclusions depend strongly on the fact that we consider only **connected** polyspheres and polyRD. If we allowed disconnected pieces, many more differences between polyspheres and polyRD would be seen.

We have now seen the two types of planar polyspheres (meaning that the sphere centers lie in a plane). Since these are directly related to polyominoes and polyhexes, there is a certain familiarity to them. The ways that they can fit together to make 3D objects are remarkable and surprising, making for excellent puzzles [1, 3, 4, 9]. Most of these puzzles are not interlocking.

Most polyspheres are truly 3D objects, and come in mirror image pairs. Even 3D pieces using 5 spheres (pentaspheres) can be very confusing. Consider that one can look at a diagram of a face-joined polycube piece, even with 20 cubes, and easily enter it into BurrTools. The same process for a general pentasphere can be baffling.

Classification and Notation

Torsten Sillke has a nice web page [8] where he lists all polyspheres composed of n spheres for n up to 5 (he has sent me additional files up to $n = 8$). He gives the integer coordinates of the sphere centers, and for each n sorts the polyspheres lexicographically, meaning that he sorts the n sphere center coordinates as a long string. This gives a well-defined ordering of polyspheres, but mixes them fairly randomly with respect to planar and non-planar types.

I present here an alternative notation where polyspheres that share similar properties are kept together. Every polysphere is classified into exactly one of the five types:

S = planar in the square geometry (like Figure 1, equivalent to a polyomino)
H = planar in the hexagonal geometry, and not X
X = planar in the hexagonal geometry, chiral made from rhombic dodecahedrons
N = non-planar and non-chiral (3D, usually with a plane or center of symmetry)
M = chiral, mirror image is different (must be 3D)

A specific polysphere is denoted by, for example, "5M12", where

- "5" represents the number of spheres (*n*).
- "M" is the piece type.
- "12" is a unique identifying number within the M pentasphere type.

The pieces are sorted by the number of spheres, then the type (in the order above) and finally by identifying number. To determine the identifying number, we could use Torsten's lexicographic ordering. However, it would be nice if the first piece "nS1" was always the linear polysphere, and this is not the case with Torsten's scheme. This piece is special because, technically, it is the only piece that belongs in both type **S** and type **H**. For the purposes of our notation, this piece will be considered to be **only** in type **S**.

I tried a number of possibilities for sorting within a type, but eventually settled on a scheme based on the principal moments of inertia of the polysphere, $\lambda_1 \geq \lambda_2 \geq \lambda_3$. The sorting is by λ_1, with λ_2 and λ_3 breaking ties. Even using all three, ties can occur. The physics-oriented reader may enjoy investigating the 12 pentominoes and find the unique pair which have the same three principal moments of inertia, and thus behave identically as rotating, rigid bodies. For such cases, final tie breaking is done using Torsten's lexicographic (alphabetic) ordering.

When a piece is chiral, the mirror image piece is denoted by using a lower case type, for example 5M12 and 5m12 are different pieces that are mirror images of one another. The distinction between **X** and **H** types is only significant for polyRD—3X1 and 3x1 are identical polyspheres, but differ as polyRD (this is the piece in Figure 2).

The total number of *n*-sphere polyspheres increases very rapidly, approximately 10-fold with each added sphere. The total number of nonospheres (*n* = 9) is over 3 million (over 99% type **M**), by comparison there are merely 48,311 nonominoes. Table 1 gives the total number of pieces of each type to *n* = 9. The row "OEIS" identifies certain sequences in the On-line Encyclopedia of Integer Sequences [10], the column "RD + Mir" counts PolyRDs including mirror images.

I have also created BurrTools files containing all polyspheres up to hexaspheres (n = 6). These files show each piece labeled using my notation, that of Torsten Sillke [8], and Ishino Keiichiro [11]. These files can be found on the CFF web site under the supplementary material for issue 81. Please download them!

The highest numbered polyspheres of type **N** are the most compact and symmetric of polyspheres. For example 4N4 is a 4-ball tetrahedron, 5N19 is a 5-ball pyramid, and 6N97 is a 6-ball octahedron.

n	S	H	X	N	M	Total	Tot + Mir	RD + Mir
OEIS	A105	H+X+1=A228				A38173	A38174	A38172
1	1	0	0	0	0	1	1	1
2	1	0	0	0	0	1	1	1
3	2	1	1	0	0	4	4	5
4	5	3	3	4	5	20	25	28
5	12	6	15	19	79	131	210	225
6	35	16	65	97	998	1,211	2,209	2,274
7	108	28	304	377	11,917	12,734	24,651	24,995
8	369	72	1,375	1,732	140,610	144,158	284,768	286,143
9	1,285	123	6,448	6,623	1,673,258	1,687,737	3,360,995	3,367,443

Table 1. Count of the number of *n*-sphere polyspheres by type (from [8]).

One advantage of this notation is that it applies to **both** polyspheres and polyRD, and it is easy to make the conceptual shift from polyspheres to polyRD. Any puzzle which contains no pieces of type **X** can be immediately converted to polyRD. Note that this only means the pieces will fit together in the final shape, it says nothing about whether the puzzle can actually be assembled from polyRD. If the puzzle contains pieces of type **X**, for each **X** piece one must either include the polyRD piece or its mirror image. If such a puzzle assembles in multiple configurations, it may not work made from polyRD, because one assembly is likely to require **X**, and another its mirror image **x**.

Pentasphere Puzzles

Like the popular pentominoes, puzzle pieces made from 5 spheres—the pentaspheres, are among the most interesting. But there are 210 pentaspheres! Tetrahedron ball stacks of height 3, 4 and 5 can be evenly divided into pentaspheres

In how many ways can a 10-ball tetrahedron be separated into two pentaspheres (identical or not)? This basic question can be answered deductively, or using BurrTools [5]. There are only five ways to do it: {5H4, 5N18}, {5H6, 5N14}, {5M66, 5M70}, {5M64, 5m64} and {5M77, 5m77}. We can see from the notation that the first two involve a planar piece, and a symmetric 3D piece, while the last two involve a piece plus its mirror image.

Perhaps the most interesting of the five is the middle one, {5M66, 5M70}. To see if these two pieces can be assembled, I glued 10 wooden balls into the final configuration. I found that the two pieces could not be separated without breaking them. This assembly is interlocking but cannot be assembled from rigid pieces. However, if the pieces are made from a sufficiently strong and flexible plastic, they snap together quite nicely.

The reader may find the level diagrams in Figure 3 rather baffling at first. Here sphere centers can occur at Cartesian coordinates where $x + y + z$ is even, and the numbers (when present) signify a sphere at this z-level. This is exactly the way these pieces can be entered into BurrTools, but is admittedly an unusual way to look at a tetrahedron (far right, Figure 3).

		0							1					0		2	1	0
1	0,2			1	0	1			0,2			1	0	1		1	0,2	1
0				0		2		0		2		0				0	1	2

5M66 5M70 5M64 5M77 10-ball tetrahedron

Figure 3. Level diagram for pentaspheres to build a 10-ball Tetrahedron.

The separation {5M64, 5m64} is also interlocking, and rigid disassembly is impossible. Even with slightly flexible pieces this puzzle cannot be assembled. Iwasawa Hirokazu uses rotating joints to make assembly possible in his “Quinto Twin” puzzle [12].

The final separation {5M77, 5m77} is not interlocking—however, it has another unusual property. It is the only one of the five for which it is possible to take two sets of pieces, and build a 20-ball tetrahedron. Moreover, using three sets of pieces, plus the X-pentomino 5S12, one can build a 35-ball tetrahedron. Finally, using 12 complete sets of {5M77, 5m77} one can build a 120-ball tetrahedron (8 balls high), and the solution is unique. Because piece 5M77 loves to stack into tetrahedrons of many sizes, I call it the “tetrahedron building block”. All the puzzles so far mentioned do not make use of a piece of type X, so can equally be made from RD. This is particularly important for this last puzzle using 5M77 and 5m77, which assembles into multiple configurations.

A second source of puzzles is to consider all ways to build the 20-ball tetrahedron from four identical or mirror image pentaspheres. This has been investigated previously for planar pieces, with some results for non-planar pieces [3].

First, consider the separation of the 20-ball tetrahedron into two identical “half-tetrahedrons” (Figure 4). Using BurrTools, this half-tetrahedron can be divided into two identical pieces using 5X11, 5X14, 5N18, 5M20 or 5M70.

To build a 20-ball tetrahedron, we can use two sets of any piece which can build the half-tetrahedron, or four of 5H4, 5H6, 5X9, 5X10, 5X12, 5M12, 5M21, 5M37, 5M46, 5M66 or 5M77, the last 6 of which require two mirror image pairs. My favorite among these is 5M12—it cannot be assembled from rigid pieces but snaps together quite tightly made from plastic. I call this puzzle “Interlocking Tetrahedron 1” [11]. This puzzle and 5M37 are closely related to Wiezorke’s elegant puzzle “Blossom” [9]. The 5M37 version assembles even from rigid pieces.

				1				1					0			1				1	0
				0				0,2				0				0			1	0	1
	0		0				0				0			1	0			1	0	1	
		0				0					1	0		0				0	1		

5M12 5M21 5M37 5X11 half-tetrahedron

Figure 4. Level diagram for pentaspheres to build a 20-ball tetrahedron.

Another interesting puzzle uses piece 5M21 (two copies plus two mirror images). This puzzle appears in Wiezorke’s Compendium [9] as “Stan’s Tetrahedron (1988)”. However, Stan Isaacs denies inventing this puzzle! This puzzle interlocks nicely and can be assembled from rigid pieces. I have made several beautiful copies out of stainless steel using Shapeways [13]. It has also been reinvented by Iwahiro Iwasawa as “Ball Pyramid Puzzle Quartet” [12].

A final nice puzzle uses four copies of the planar polysphere 5X11. This can be solved in a simple way by building two half-tetrahedrons, or more interestingly in an interlocking assembly. Stan Isaacs told me he remembers discovering this interlocking solution around 1988. When made from PolyRD, this puzzle must be made using two copies of 5X11 and two copies of its mirror image 5x11.

Polysphere puzzles can be produced in plastic or metal using 3D printing technology. BurrTools [6] exports directly to STL format for use in a 3D printer. Many of the puzzles above can be purchased from the 3D printing company Shapeways [13], or you can easily devise and print your own puzzle designs.

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