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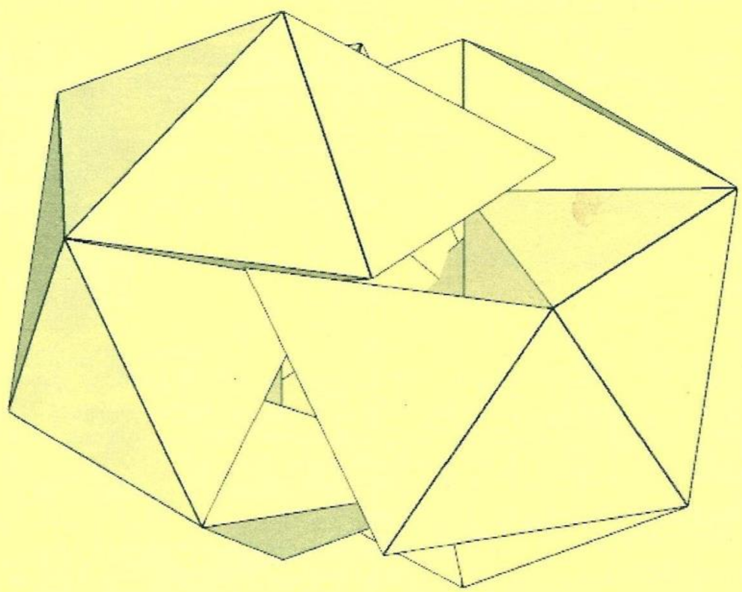

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MARCH 2012

CFF, Hilvoordestraat 14, 2284 BK Rijswijk

More Icosahedron Puzzles

by George Bell

Introduction

In CFF 50 [1], Wayne Daniel introduced a fascinating new puzzle based on a dissection of an icoashedron into 40 non-regular tetrahedra. In this article we will revisit the geometry of this puzzle. We will find some new piece shapes and new puzzles, including a coordinate motion puzzle where the fit of the puzzle can be controlled by one parameter. We use the same notation as Wayne Daniel [1].

Puzzle geometry

We first cut an icosahedron into 20 identical "face tetrahedra" by drawing radial lines from the centre to each vertex (Figure 1). Each face tetrahedron contains one equilateral face of the original icosahedron, and three identical isosceles faces. We then subdivide each face tetrahedron into inner and outer tetrahedra by a plane going through the outside edge **e** and a point **p** between the centre **c** and **v**.

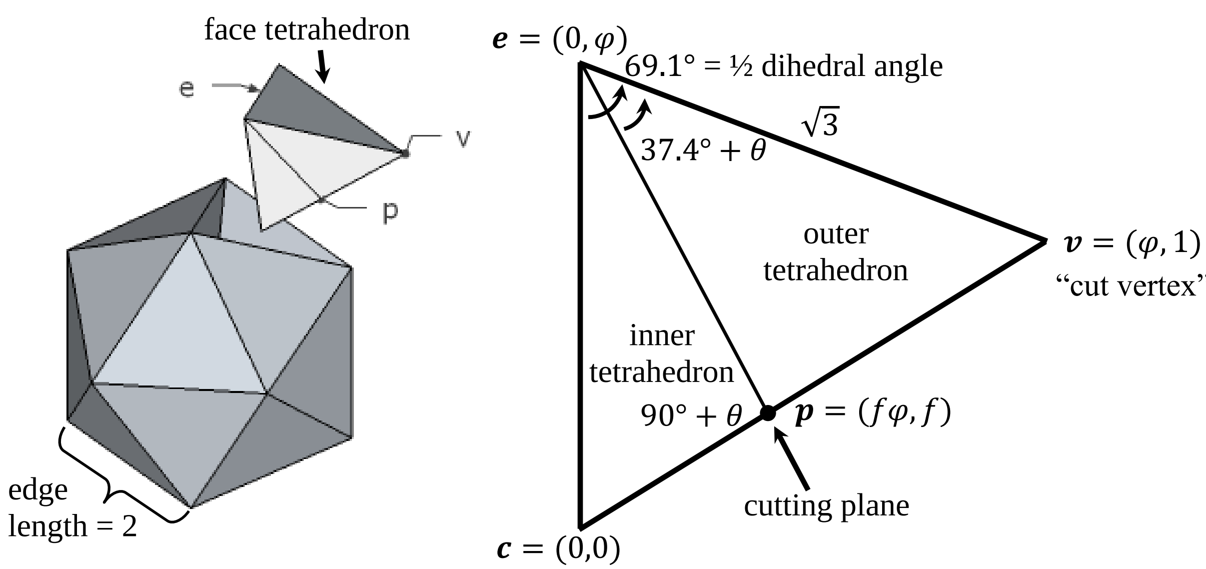

Figure 1. An icosahedron dissected.

If we view the face tetrahedron projected into the plane perpendicular to the edge **e**, we get the diagram in Figure 1 (right), where $\varphi = \left(1 + \sqrt{5}\right)/2$ is the golden ratio. The location of the cut can be specified by the angle θ or the parameter *f*—the fractional distance from the centre of the icosahedron to **v,** the **cut vertex**. Wayne Daniel chose the cut so that the line **e**-**p** is perpendicular to the opposite edge **c**-**v**, this corresponds to the choice of $\theta = 0$ or $f = \sqrt{5}/5$. In general θ can be positive, negative or zero. The simplest formula I have found which relates *f* and θ is:

$$\sin\theta = \left(\frac{\sqrt{5}}{5} - f\right)\sqrt{\frac{\varphi + 2}{(f\varphi)^2 + (\varphi - f)^2}}$$

Puzzle pieces

We now glue these 40 tetrahedra together to make puzzle pieces. Wayne Daniel devised a clever way to do this. He considered four connected faces of an icosahedron. There is essentially only one way to do this where the starting and final faces do not share a vertex. For example, in Figure 2, consider the faces {1, 2, 8, 9}. We make the puzzle piece from the outer tetrahedra for faces 1 and 9, and the inner tetrahedra for faces 2 and 8. It takes 10 such puzzle pieces to make a full icosahedron. The reason this joining is so clever is that each piece is held in place by the outer tetrahedra of two other pieces covering its middle two faces. In other words the puzzle is automatically interlocking! What is not clear is if these puzzles can be assembled, we will return to this point later.

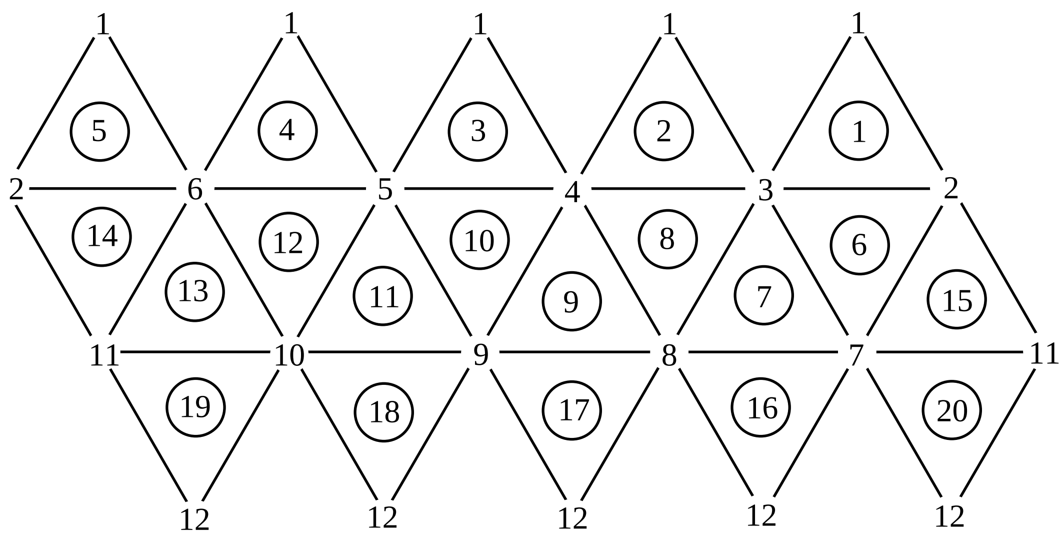

Figure 2. Icosahedron net with faces and vertices numbered (same as [1]).

We now consider how these puzzle pieces could fill an icosahedron. First, note that any piece is either right-handed (for example if made from faces {1, 2, 8, 9} or {1, 6, 15, 20}) or left-handed ({2, 1, 6, 15} or {1, 6, 7, 16}). As Wayne Daniel noticed, what we need is a list of 10 sets of four adjoining faces, such that each face appears exactly once at the end of each set (outer tetrahedra) and exactly once in the middle of each set (inner tetrahedra). The fourpossible arrangements are given in Table 1.

	Arrangement #0	Arrangement #1	Arrangement #2	Arrangement #3
1	1, 2, 8, 9 (R)	1, 2, 8, 9 (R)	1, 2, 8, 9 (R)	1, 2, 8, 9 (R)
2	2, 3, 10, 11 (R)	2, 3, 10, 11 (R)	2, 3, 10, 11 (R)	2, 3, 10, 11 (R)
3	3, 4, 12, 13 (R)	3, 4, 12, 13 (R)	3, 4, 12, 13 (R)	3, 4, 12, 13 (R)
4	4, 5, 14, 15 (R)	4, 5, 14, 15 (R)	4, 5, 1, 6 (L)	4, 5, 1, 6 (L)
5	5, 1, 6, 7 (R)	5, 1, 6, 7 (R)	5, 14, 13, 19 (R)	5, 14, 15, 20 (L)
6	18, 17, 9, 8 (R)	6, 15, 20, 19 (L)	7, 6, 15, 14 (L)	7, 16, 20, 19 (R)
7	19, 18, 11, 10 (R)	8, 7, 16, 20 (L)	8, 7, 16, 20 (L)	8, 7, 6, 15 (R)
8	20, 19, 13, 12 (R)	10, 9, 17, 16 (L)	10, 9, 17, 16 (L)	10, 9, 17, 16 (L)
9	16, 20, 15, 14 (R)	12, 11, 18, 17 (L)	12, 11, 18, 17 (L)	12, 11, 18, 17 (L)
10	17, 16, 7, 6 (R)	14, 13, 19, 18 (L)	15, 20, 19, 18 (R)	14, 13, 19, 18 (L)

**Table 1. Face numbers in ten overlapping groups of four,
which cover an icosahedron (with left (L) and right (R) pieces identified).**

In Wayne Daniel's original article [1], this table was incorrect, confusing many of us who attempted to build these puzzles. Wayne Daniel sent us a corrected table, and our Table 1 can be considered a correction for Table 1 in the original article. Wayne Daniel wrote that any solution must use 5 right-handed and 5 left-handed pieces. Stephen Chin was the first to notice that this was not the case. I wrote a program to search for all distinct arrangements which include the piece {1, 2, 8, 9}, and it finds only the four listed in Table 1, including the new Arrangement #0 which uses all right-handed pieces.

Arrangements #0 and #1 are very regular, with pieces arranged symmetrically with respect to the vertical axis in Figure 2. A subtle point is that arrangements #2 and #3 are actually mirror images of one another. The pieces in #2 that are mapped to #3 after reflection are (1,2,3,4,5,6,7,8,9,10) ↔ (8,9,10,6,5,7,1,2,3,4). In Table 2, the six pieces in the "standard" locations (of arrangement #1) are shaded.

To fully specify any right-handed puzzle piece, we need to list the sequence of cut vertices for the four faces {1, 2, 8, 9}. Since there is a choice of 3 vertices for each face, it would appear that there are 3^4= 81 possible piece shapes. However, many of these pieces are not connected or are identical. In Table 2, we give the cut vertices for the 10 connected pieces—each has a different mirror image, so 20 pieces total. Some pieces can be rotated "end for end" (flipped) and they are unchanged, these pieces are called symmetric.

The pieces are labeled by capital letters, with alternate letters A, C, E, G, … right-handed pieces, and piece B is the mirror image of A, etc. This is Wayne Daniel's notation, but he listed only 12 unique pieces (A-L, see Figure 4 in [1]). The pieces M-P he did not consider because the two middle cuts are made along the same vertex. Pieces Q-T have a more serious flaw in that they actually contain holes. If these pieces are used in a puzzle, other pieces must connect through the holes and the pieces cannot be separated. We will find no further use for pieces Q-T.

Piece	Cut Vertices	Flipped	Piece	Cut Vertices	Flipped
A	1, 4, 3, 8	symmetric	K	3, 1, 8, 4	symmetric
C	1, 4, 3, 4	3, 4, 3, 8	M	3, 4, 4, 8	1, 3, 3, 4
E	3, 4, 3, 4	symmetric	O	1, 4, 4, 8	1, 3, 3, 8
G	1, 4, 8, 4	3, 1, 3, 8	Q	1, 3, 4, 8	symmetric
I	3, 4, 8, 4	3, 1, 3, 4	S	1, 3, 8, 4	3, 1, 4, 8

Table 2. Definitions of the ten right-handed pieces over faces 1, 2, 8, 9.

Puzzles with ten identical pieces

These puzzles must use Arrangement #0 in Table 1. I wrote a program to solve these puzzles, it reports that only pieces I and M work with 10 identical copies (plus, of course their mirror images J and N), see Figure 3. The 10xI or 10xJ puzzles were discovered by Stephen Chin, he also discovered that the fit of this puzzle can be adjusted by changing the angle θ.

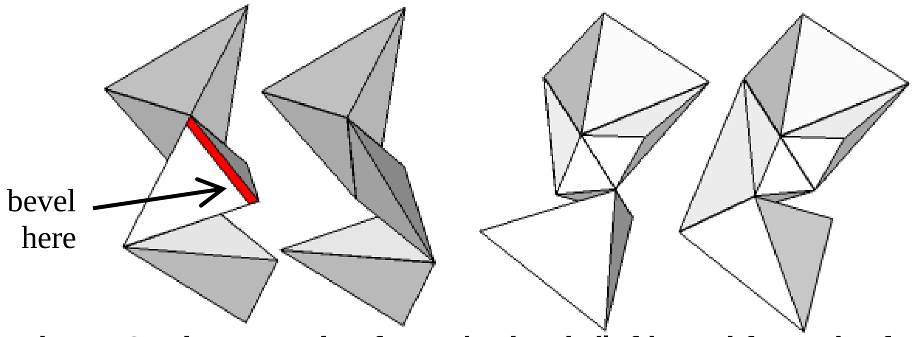

Figure 3. Piece I and M from the back (left), and from the front (right). The center of both pieces has been cut out, leaving a small icosahedron.

This puzzle must be disassembled using coordinate motion, and Stephen Chin likes to adjust his puzzles so they will spin on a table for a second or two, then suddenly explode into 10 pieces. The reason for the delay is that it can take time for the spin axis to coincide with the disassembly axis for the coordinate motion. After many prototypes, he came up with the magic angle $\theta = 4.6°$ or $f = 0.389$, which he uses in his 10xJ icosahedron puzzles. He also adjusts the fit of his puzzles by beveling down the region identified in Figure 3.

Because most cutting planes for this puzzle pass through the origin, these puzzles can be modified so that the assembled shape is any object with icosahedral (or higher) symmetry. Indeed, we can also make the puzzle hollow by cutting out an inner icosahedron, or in general anything with icosahedral symmetry. The easiest way to do this is to cut out an icosahedron from the centre of size **f** times the outer icosahedron (Figure 3).

Stephen Chin put the assembled puzzle in a lathe and shaved the icosahedron into a sphere (Figure 4), football, bomb and apple. Because the latter two do not have icosahedral symmetry, the pieces are no longer identical. Turning a puzzle in a lathe which comes apart when spun presents great challenges, and the puzzle must be glued together (temporarily) for this step. Even the sphere pieces in Figure 4 do not end up entirely identical, and must be reassembled in the right order for a perfect fit. The apple version of this puzzle “1 Pinko Ringo” was one of the top vote getters in the 2010 IPP design competition [2].

Figure 4. Stephen Chin’s 10xJ puzzle, with sphere external form.

Because this puzzle is so difficult to make in wood, it is ideal for 3D printing. I prefer the sphere versions because this removes most of the sharp edges on the pieces. The pieces must slide easily against one another during assembly—many 3D printed materials are too rough and will require sanding or polishing. To explore the effect of changing the parameter θ on the fit, I printed five sphere versions of this puzzle from $\theta = 0.6°$ to $8.6°$ in $2°$ intervals. At $\theta = 8.6°$, the puzzle goes together easily and falls apart just as easily when spun. As θ is decreased, the puzzle becomes tighter and more difficult to assemble, until near $\theta = 0°$, it appears impossible to assemble from rigid pieces. The angle for the best delayed "explosion" is between $2.6°$ and $4.6°$ [3].

We note that the magic angle $\theta = 4.6°$ applies only to the 10xI (or 10xJ) puzzles. The 10xM puzzle is much easier to assemble or take apart, and the fit does not change much with θ.

Puzzles with different pieces

A useful concept for these puzzles is a **subassembly**—this is simply some subset of the full icosahedron that the puzzle pieces can fill, usually several different ways. For example, we can assemble 2xI to form S1 in Figure 5. Five copies of S1 fill the icosahedron, and this is one way to assemble the 10xI puzzle. We can also make S1 from 2xM. This tells us that 8xI + 2xM make a puzzle, and so on. Two copies of S1 can be nested together, and can be filled by 2xG + 2xK. Thus, we can make a puzzle from: 2xG + 2xK + 4xI + 2xM. Three copies of S1 can be filled by 2xI + 2xK + 2xO. All these combinations give us many puzzles involving pieces G, I, K, M and/or O, all in Arrangement #0 (see Table 3).

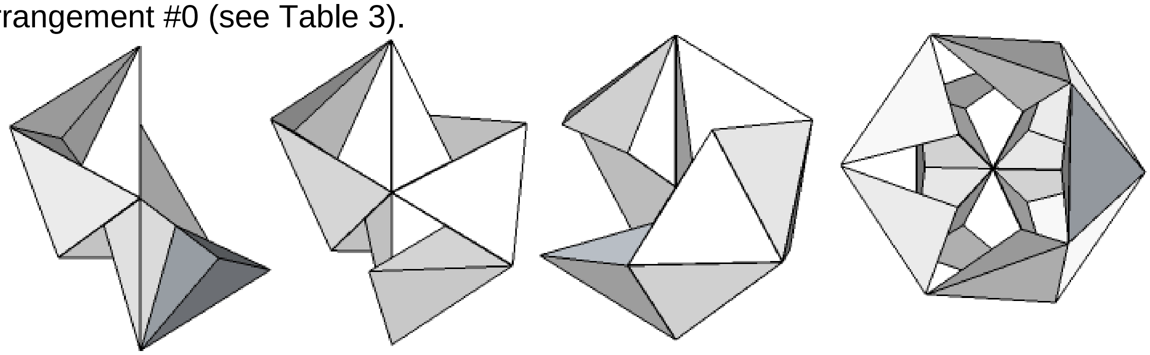

Figure 5. Subassemblies S1, 2xS1, 3xS1, and S2

The subassembly S2 is mirror symmetric and can be made from 5xI. Therefore we can also make S2 from 5xJ, this gives the puzzle 5xI + 5xJ in Arrangement #1, listed in [1]. A similar subassembly can be found for piece C, with the critical difference that it is not mirror symmetric. To complete the puzzle requires 5xD, also in [1].

Arrange-ment	Number of different piece shapes in puzzle										Total
	1	2	3	4	5	6	7	8	9	10	
0	2	4	10	17	8	0	0	0	0	0	41
1	0	2	0	8	0	8	0	0	0	0	18
2	0	0	0	4	20	82	104	122	30	4	366

Table 3. The number of puzzles for each arrangement using pieces A-P.

The puzzles presented so far are relatively easy, using the simplest arrangements #0 and #1. More difficult versions of this puzzle use arrangement #2 or #3, and have as many pieces different as possible. My program finds that pieces M-P can never appear in arrangements #1, #2 or #3, and that no puzzle with more than 6 different piece types has a unique solution. In Table 3, arrangement #3 has exactly the same counts as Arrangement #2 by taking the mirror image of each puzzle.

Pieces	Arr.	1	2	3	4	5	6	7	8	9	10
A+2x(B+G+I+J)+L	3	A	I2	G1	B	J1	G1	I2	B	J2	L
B+2x(G+I+J+L)+K	2	I2	G1	K	L	G2	J1	B	J2	L	I1
A+B+C+D+F+	2	G2	G1	I1	H1	C1	D2	B	F	J2	A
2xG+H+I+J	2	A	C2	I2	J2	G2	F	B	D1	H1	G1
C+D+F+2xG+	3	C1	I1	K	D2	F	G1	G2	H2	J2	L
H+I+J+K+L	3	C2	G2	G1	H1	J1	K	I1	D1	F	L

Table 4. All solutions for 4 puzzles with six or nine different pieces.

In Table 4, I show two puzzles with 6 piece types with unique solutions, and two puzzles with 9 piece types with only two solutions. In Table 4, the non-symmetric pieces have a trailing 1 or 2. This indicates orientation, for example “G1” means piece G with face vertices and cut vertices in the order given in Tables 1 and 2. “G2” indicates that the piece must be flipped end for end, so use the “Flipped” column in Table 2. In the original article [1], Wayne Daniel gives the four puzzles with 10 different pieces, each has either 8 or 12 assemblies.

Any assembly using arrangement #2 or #3 can be separated into two non-identical halves, S3 and S4, as shown in Figure 6. The halves are somewhat reminiscent of a Pennyhedron [4]. The partition of pieces numbered as in Table 1 is S3={1, 2, 7, 8, 9} with S4 consisting of the other five pieces (a second alternative is S3={1, 2, 3, 8, 9}). Unlike a Pennyhedron, the two halves do not simply slide together. Assembly requires coordinate motion where multiple pieces are shifted slightly, this does not depend much on the angle θ. If we remove piece 3 or 7, the remaining halves do slide together as rigid bodies. A quick spin and the puzzle usually separates into these halves, from a pure fit perspective these puzzles are much easier to assemble and disassemble than the 10xI version.

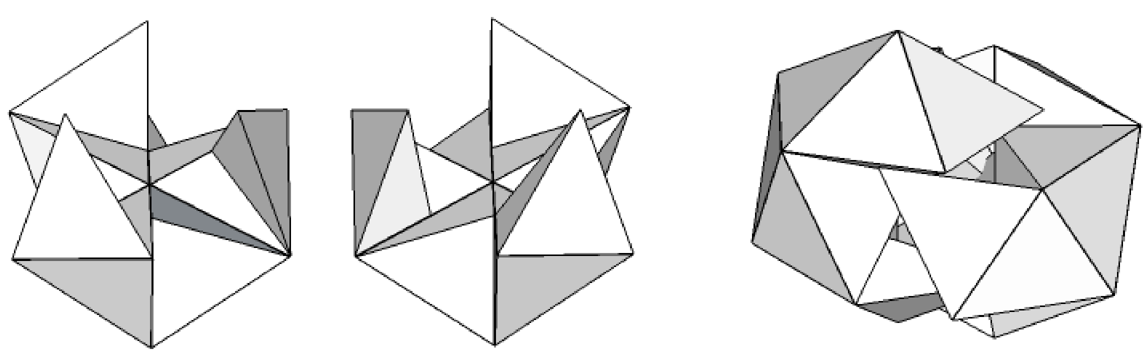

Figure 6. Separation of Arrangement #2 into non-identical halves, S3 and S4.

A nice set of pieces for exploring this geometry is one each of the 12 pieces A-L (Figure 7). Using this set one can explore all of the 10-piece puzzles in [1]. With an extra G, the two 9-piece puzzles in Table 4 can be assembled, and with additional B, I, J and L all of the Table 4 puzzles can be assembled. In Figure 7, the assembled form is an edge-beveled icosahedron with an internal hollow icosahedron [3].

Figure 7. Pieces A-L with $\theta = 0$ shown alphabetically (left, same layout as Figure 4 in [1]), assembled (right).

References

[1] Wayne Daniel, Some Icosahedron Puzzles, CFF50, part 3, pp. 13-17.

[2] 2010 IPP design competition, http://www.puzzleworld.org/DesignCompetition/2010/results.htm

[3] Shapeways shop, Poly Puzzles, http://www.shapeways.com/shops/polypuzzles

[4] Stewart Coffin, Geometric Puzzle Design, A K Peters (2009) 136-138.