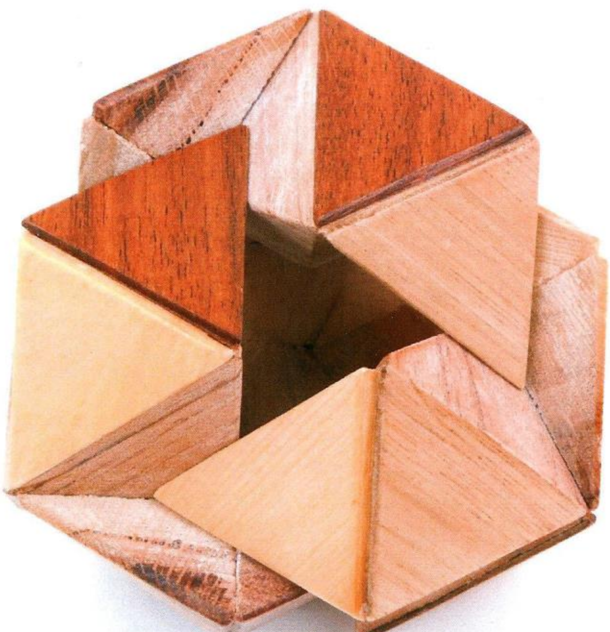
CFF
FOR FUN
CUBISM



93

MARCH 2014

CFF, Hilvoordestraat 14, 2284 BK Rijswijk

The *Pennyhedron* Revisited
by George Bell

Introduction

The *Pennyhedron* is a two-piece puzzle invented around 1971 by Stewart Coffin and his children (!) [1; 136-8]. The original version (Figure 1, right) has the maddening property that if you grab the assembled puzzle between two fingers of each hand, opposing fingers will always touch both pieces, and it will never come apart. Separating the pieces requires an unnatural, three-fingered grasp, and we refer to this version as a *3-Finger Pennyhedron*. Coffin named it a *Pennyhedron* because some of the first versions were hollow, with a penny inside.

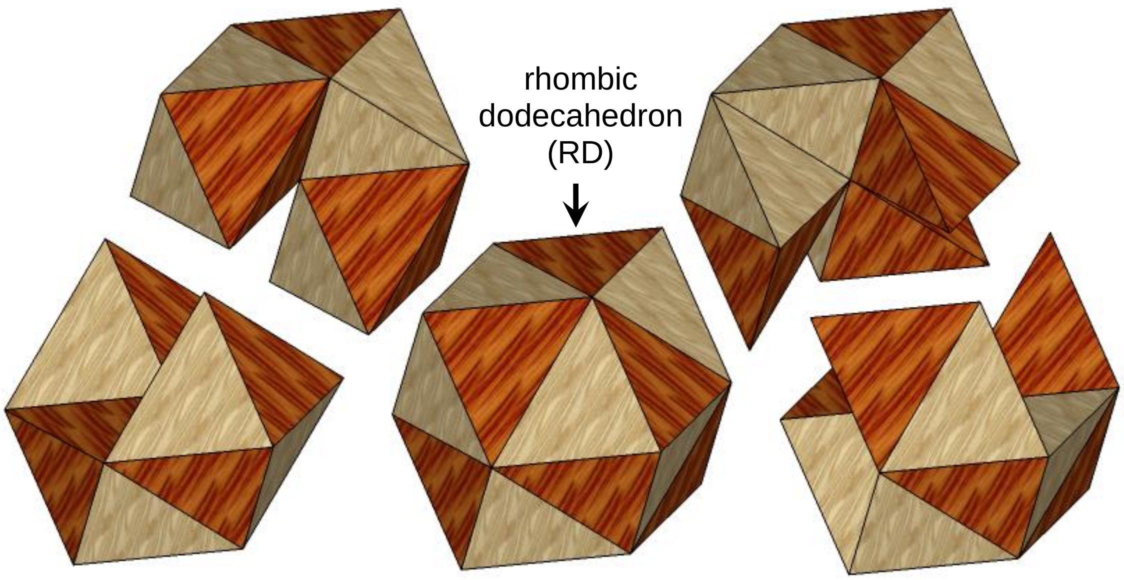


Figure 1. *Pennyhedron Tricky Pair* (design #52C by Stewart Coffin)

Coffin initially considered the *Pennyhedron* an amusing toy—too simple to be a puzzle. However, it proved surprisingly popular as a puzzle. One of the most popular versions he calls *Pennyhedron Tricky Pair* [1; 137-8] (Figure 1). Both two-piece puzzles assemble into a rhombic dodecahedron (RD), but in very different ways. The pair on the left comes apart easily using a two-fingered grasp and we refer to it as a *2-Finger Pennyhedron*.

From the beginning, the *Pennyhedron* did not refer to a single puzzle design, but a whole class of designs. Coffin's design #52 includes six *Pennyhedron* variations [2; 52-3]. In this article, a *Pennyhedron* is an interlocking dissection of the rhombic dodecahedron (RD) into two, three, or four pieces.

For IPP33, Stephen Chin created several sets of around 15 different *Pennyhedra*. Watching attendees amuse themselves figuring out how to take each RD apart inspired me to undertake a systematic search for all *Pennyhedra* designs using BurrTools [4]. This article presents my findings.

Experienced puzzlers now recognize the RD and immediately try the three-fingered grasp. One way to confuse them is to change the external shape. Any shape with octahedral symmetry will work including a cube (Figure 2), octahedron (see [2; 53]), cuboctahedron (the dual of the RD) as well as any RD stellation. Another interesting option is the Stienmetz Solid or Tricylinder (Figure 2), the solid formed by the intersection of three cylinders at right angles. This is a geometrical situation where the RD appears naturally—it has curved faces and a rotund, overinflated look. Using color 3D printing, I have made a *3-Finger Pennyhedron* in the form of a Tricylinder [3], jokingly referred to as a *Pillowhedron* (Figure 2). Although the pieces appear to screw together, they actually slide together without rotation. Any *Pennyhedron* design can be adapted to the above external shapes, and can be solid or hollow.

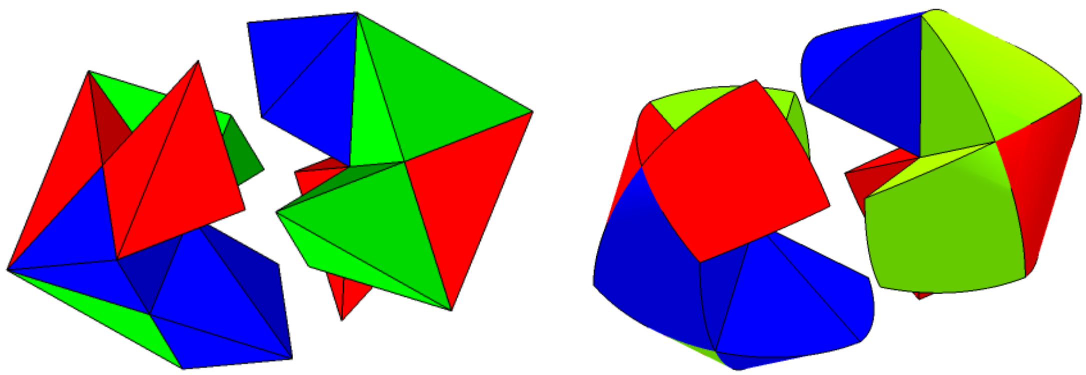


Figure 2. A *3-Finger Pennyhedron* disguised as a cube and a *Pillowhedron.*

Puzzle geometry

We consider the RD divided into 24 tetrahedra (tetras) by dividing each face in half and drawing radial lines from each vertex to the centre. Coffin calls these "tetrahedral blocks" [1; 85, 103]. BurrTools [4] further subdivides each tetra into two equal pieces, so a tetra is a two-voxel piece in the "Rhombic Tetrahedra" geometry. In Figure 1, the tetras are made from alternating wood types -- recommended for maximum obfuscation. When any *Pennyhedron* is expertly made in this two-wood pattern, the divisions between pieces are virtually undetectable.

Suppose we dissect an RD into two pieces made from tetras, and that each piece is connected. If we try to separate these two pieces, one of three things can happen:

1. The two pieces cannot be separated, both pieces are Unremovable (U).
2. The two pieces can be separated in one direction only, called Slide Out (S).
3. The two pieces can be separated in more than one direction, Fall Apart (F).

Table 1 contains counts of piece types for pieces made from 5-19 tetras. Because pieces are classified in pairs, the row for 5 tetras has the same counts as the row for 24–5=19 tetras.

Number of Tetras	Number of pieces of type U	S	F	Total
5 or 19	0	0	8	8
6 or 18	1	3	17	21
7 or 17	2	11	22	35
8 or 16	12	28	37	77
9 or 15	40	52	42	134
10 or 14	108	82	56	246
11 or 13	190	104	57	351
12	244	119	64	427

Table 1. Piece counts by type (U=Unremovable, S=Slide Out, F=Fall Apart).

One thing we have glossed over is that normally (for example, with polycubes) whether a piece can be removed depends not only on the piece itself, but also on its orientation and location within the puzzle. However, these tetra puzzle pieces are unusual in that there is only *one location* each can be placed within the RD (up to symmetry). This is because each tetra goes all the way to the centre of the RD. So we can think of the classification (U, S or F) as a property of the piece itself.

Ultimately, we are interested in whether or not a puzzle is interlocking. Even for two-piece puzzles this is not a simple concept! Informally, a puzzle is interlocking if the pieces hold one another in place. The main clue that a two piece puzzle is *not* interlocking is that the pieces can move apart in more than one direction, or in more than one way. Pieces of type F are not useful for creating interlocking puzzles, but as we will see, type S (and even U) pieces may or may not be interlocking. One reason is that the classification above does not include twisting as a separation method. Really, the only way to determine if a puzzle is interlocking is to make a physical copy.

The figures below contain two-color diagrams. If it is not clear what puzzle piece is depicted, the reader should download the BurrTools files which are in the CFF download page for this issue. If a puzzle is listed with no inventor, it is due to the author and Stephen Chin.

Two-Piece *Pennyhedra*

Two 12-tetra pieces must be of type S to interlock, and from Table 1 we see that there are 119 such pieces. There are 54 pairs of different pieces which form an RD, plus 11 pieces which can pair with themselves or their mirror image. Figure 3 shows the three pieces which can be paired with their mirror image. They include the familiar 3-Finger piece, plus two variations which can be obtained by moving one tetra. The black arrows indicate which tetra is moved to make the piece to the right.

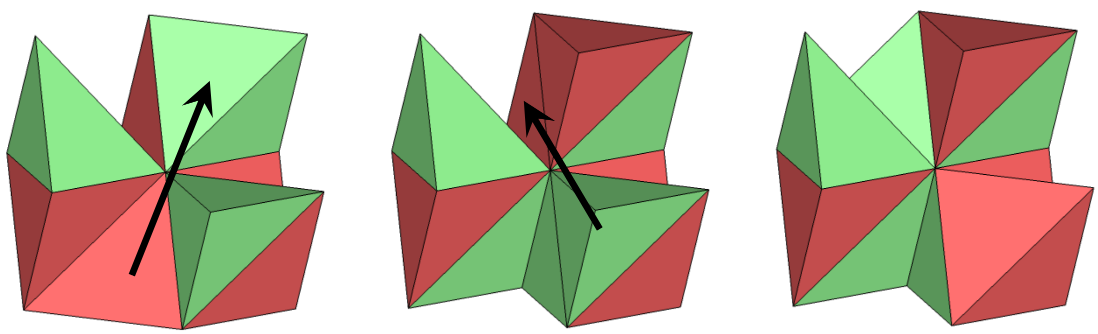


Figure 3. Standard *3-Finger* (Coffin), *Deep-cut 3-Finger* and *2½-Finger* pieces

Figure 4 shows three type S pieces which in identical pairs make an RD. As two *Pinwheel* pieces come apart, they are free to rotate slightly around the axis of separation. This extra movement means that the pieces are not interlocking.

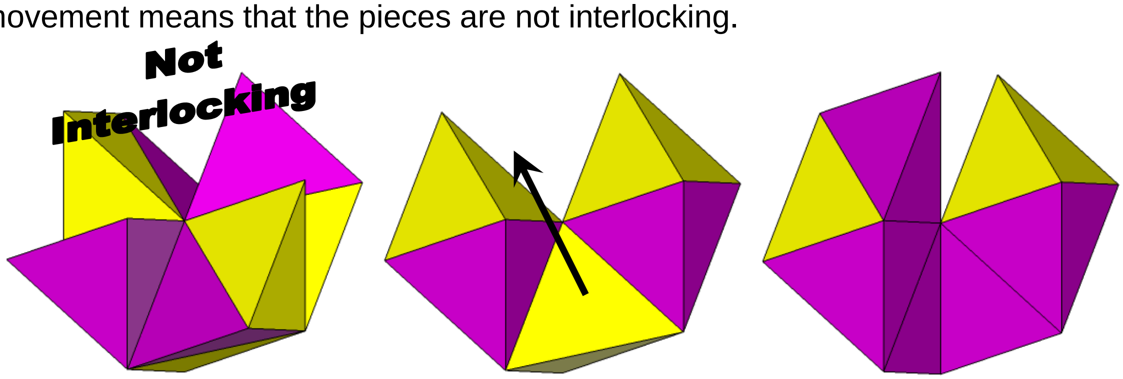


Figure 4. *Pinwheel*, *Standard 2-Finger* (Coffin) and *Deep-cut 2-Finger*

Many pieces have a "deep-cut" variation, where a "prong" sticking into one piece is extended, and the corresponding indentation enlarged.

Figure 5 shows the *Zig-Zag* piece, which appears in Coffin's writings [2; 52]. The *Zig-Zag Pennyhedron* is one of the more difficult two-piece *Pennyhedra* to open, because the usual two-fingered grip can only touch the same piece along half a face. While *2-* and *3-Finger* pieces are composed of six tetras of each color, a *Zig-Zag* piece is made from eight tetras of one color and four of the other. The *Zig-Zag* piece has several variations, one a cross between a *2-Finger* and *Zig-Zag*, and the other a deep-cut version (Figure 5). All join with a copy of themselves.

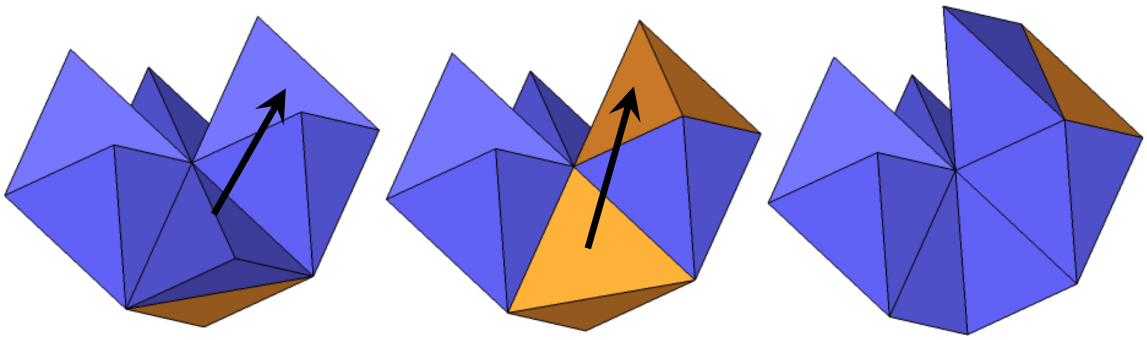


Figure 5. *Zig-Zag* (Coffin), *Hybrid 2F-Zig-Zag* and *Deep-cut Hybrid*

In Figures 3-5, we see 9 of the 11 pieces which can combine with identical or mirror image copies to form an RD. The remaining two are *Chaos* (Figure 6) which has no symmetry and is unrelated to anything seen previously, and one more piece which is not shown because it does not interlock.

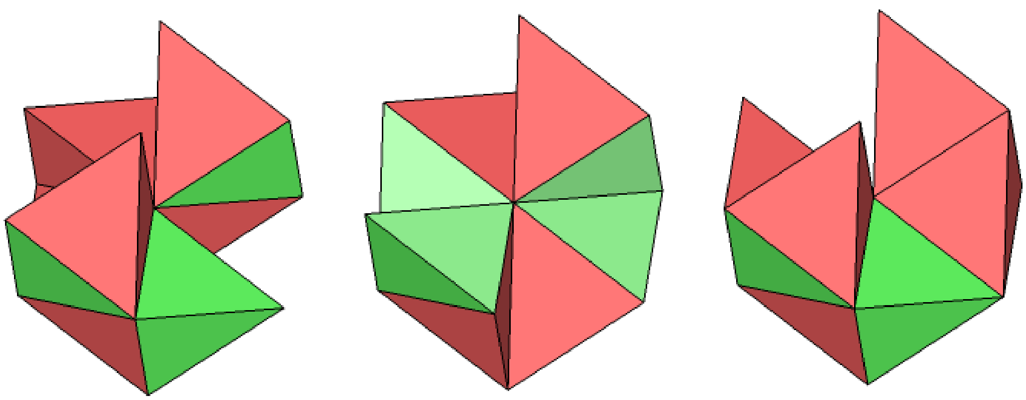


Figure 6. *Chaos*, *Alternative 2-Finger*, and *Split-tail 2-Finger*

Of the interlocking pieces presented so far, only three are symmetric. The *3-Finger* piece is three-fold symmetric, and the *2-Finger* piece and *Zig-Zag* each have two planes of mirror symmetry. Out of 54 pairs of different pieces which make an RD, two are mirror symmetric. In Figure 6 we show one piece from each of these two pairs: *Alternative 2-Finger* and *Split-tail 2-Finger*. The *Alternative 2-Finger* is totally different from a normal *2-Finger* piece, but opens using the same technique.

In all these two-piece puzzles, we have used 12-tetra pieces. Are there any interesting two-piece *Pennyhedra* where the tetras are split unequally? There are five more designs where the pieces have mirror symmetry—all of them can be obtained by starting with a symmetric puzzle (*2-Finger*, *Zig-Zag*, and *Alternative 2-Finger*) and moving one or two tetras from one piece to the other.

Three-Piece *Pennyhedra*

First, let us look for puzzles with three "unremovable" eight-tetra pieces (type U). This may sound pointless, but remember that the unremovable classification considers removing one piece when the rest of the puzzle is fixed. Coordinate motion (when three or more pieces move simultaneously) is very common in this geometry, so what we are doing is looking for coordinate motion (CM) puzzles.

BurrTools finds three different three-piece *Pennyhedra* with all type U pieces. The first uses three copies of the *Hole-In-One* piece shown in Figure 7. This puzzle does indeed come apart by CM, however as the pieces separate they are free to move along the CM axis. This extra direction of movement should lead us to suspect that the puzzle may not be interlocking, and indeed a physical copy confirms that the puzzle is loose and falls apart easily. This is an example of a puzzle with all pieces of type U which is not even interlocking!

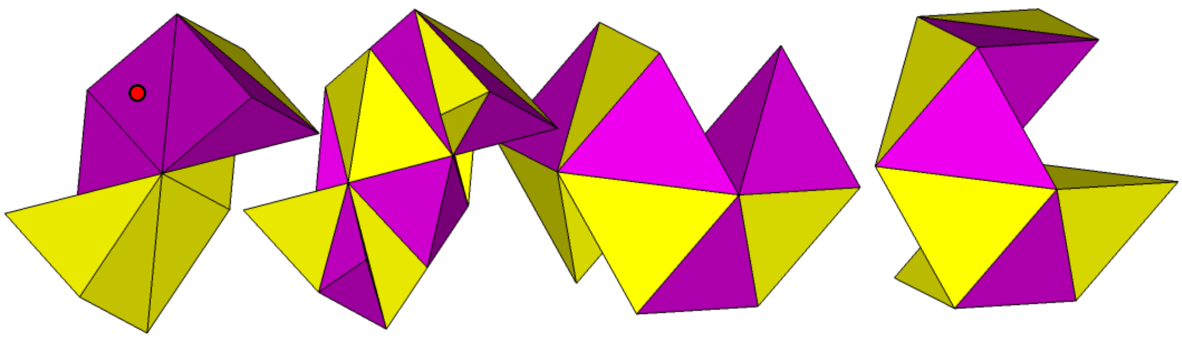


Figure 7. *Hole-In-One* (Coffin), *12 Diamond* (Nagata), *Rose* (Chin) and *Orchid*

Coffin was able to address this looseness by adding a pin to one piece and a hole to another (at the red dot in Figure 7). His cleverly named *Hole-In-One* puzzle then becomes interlocking. It is Coffin design #52A [2; 53].

Edi Nagata started with the *Hole-In-One* piece and modified it by removing and adding material along cuts outside the tetra geometry. The resulting piece (Figure 7) is no longer composed of tetras, but has the same exterior faces and volume as a *Hole-In-One* piece. His puzzle, *12 Diamonds* uses three identical pieces and interlocks into an RD via "standard" CM. *12 Diamonds* was Edi Nagata's IPP33 exchange puzzle (the modified pieces would be difficult to make in wood, so it was 3D printed).

The second puzzle uses three copies of the *Rose* piece (Figure 7), and was found in 2012 by Stephen Chin. He calls the puzzle a *Rose Pennyhedron*. This is a very interesting puzzle; I believe it cannot be assembled from rigid pieces. But since all wood flexes slightly, it does go together, and when made precisely the three pieces come together forcefully with a loud snap. The interior edges of the pieces should be rounded, otherwise they are under such stress that they may break. This is one *Pennyhedron* that can't be made hollow; it's also one of the more difficult *Pennyhedra* to take apart if you don't know its secret (the name is a hint). To help people at IPP33, Stephen Chin made a special "*Rose Pennyhedron* for Dummies" with a few tetras identified by a dark wood color.

The third puzzle is called an *Orchid Pennyhedron*, it uses two copies of the *Orchid* piece plus one *Hole-In-One* piece (Figure 7). It is the only puzzle of the three that does not use all identical pieces, and also the only one which goes together with "standard" CM (and no piece modification). The non-identical pieces have less symmetry and can bind up if you pull on two pieces, but it will always come apart if you grab each piece on opposite ends and pull (easy, if you have three hands)!

Next, let us look for puzzles with two U pieces and one S piece. The S piece will be the "locking piece" which must be removed first. BurrTools finds three such designs using three eight-tetra pieces. The first was discovered by Coffin. He called it a *Three-Piece Pennyhedron*, design #52 (again!) [2; 52]. It uses three dissimilar pieces, and we call all three of these designs *Three Dissimilar Piece Pennyhedra*, or *3DP1* (Coffin's design), *3DP2* and *3DP3* for short (Figure 8). Note that *3DP2* and *3DP3* share one piece, the middle piece in each puzzle in Figure 8, and that each piece is made from four blue tetras and four orange tetras.

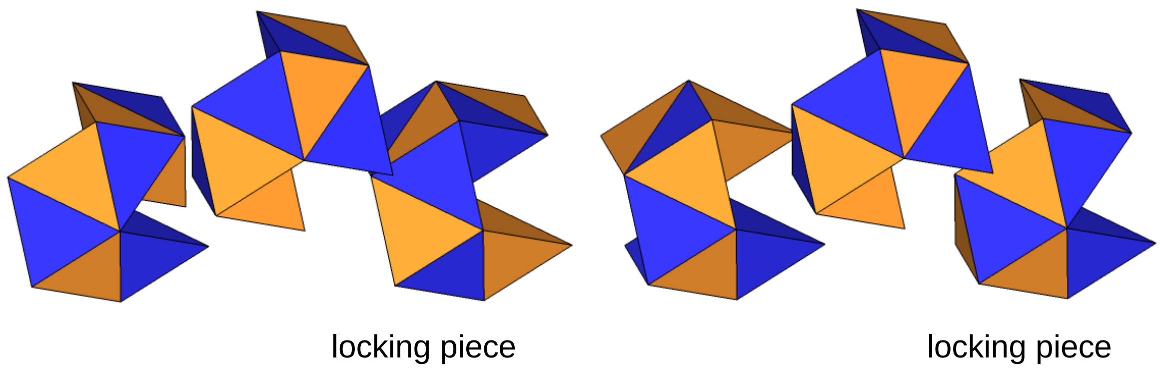


Figure 8. The three pieces of 3DP2 (left) and 3DP3 (right)

In Coffin's design *3DP1*, the locking piece contains two opposite faces of the RD. To remove it, you simply grasp the locking piece by these two faces and pull in the correct direction. Naturally, if you cannot identify the locking piece, disassembly can be difficult. However, *3DP2* and *3DP3* are even trickier because the locking piece *does not* have two opposite faces to grab. They are among the most difficult *Pennyhedra* to disassemble.

So far, all the puzzles in this section use three eight-tetra pieces. What other puzzles can we find if we use three pieces with a different number of tetras? As far as coordinate motion goes, there is nothing else. But using two U pieces and one S piece, BurrTools finds 26 additional *3DP* designs (included in the BurrTools files under the CFF download page for this issue). We note that not all of them are interlocking, and others are variations of puzzles presented in this section.

Four-Piece *Pennyhedra*

With all the three-piece puzzles we have found, it may come as a shock to learn that there are no four-piece *Pennyhedra* made from tetras! Why? From Table 1, we see that there are no U or S pieces with fewer than six tetras. Therefore, a four-piece interlocking puzzle can only use the four U or S six-tetra pieces (plus their mirror images). BurrTools quickly confirms that there is no combination of any number of these pieces which forms an RD.

To find a four-piece *Pennyhedron*, we must go to a finer dissection of the RD. The first four-piece *Pennyhedron* was introduced in 2007 at IPP27 in Australia by Stuart Gee. Stuart Gee and Stephen Chin subsequently refined the design, and it was Stephen Chin's IPP33 exchange puzzle, *De Doe Dak Ka* (Figure 9). To make this puzzle, you must cut each tetra in half: each piece is made from twelve half-tetras (or three tetras and six half-tetras). The puzzle consists of four identical pieces with three-fold symmetry, and assembles via “standard” CM. This puzzle is most easily disassembled by spinning it.

The coordinate motion planes in this puzzle are the faces of an internal cube. Therefore, if the external shape is cut down to a cube, and an internal cube is also removed, rather amazingly the puzzle becomes a cubical box. In 2011, I started from De Doe Dak Ka and designed *The Dice Box* (Figure 10); it looks like a different puzzle, but is internally identical. *The Dice Box* was 3D printed by Scott Elliott. He figured out how to print each piece flat, with living hinges you can fold along to form the final puzzle piece, held together by 3D-printed snaps.

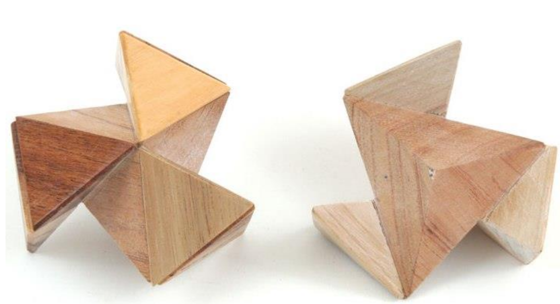


Figure 9. *De Doe Dak Ka*
(photos courtesy Nick Baxter)

By shifting the external triangular plates, it is possible to create versions of *The Dice Box* (or *De Doe Dak Ka*) where the pieces are no longer identical. For example, there is a version of the puzzle with three different piece types, with two copies of one piece needed (Figure 10, right). It assembles into a cube using the same CM.

Figure 10. *The Dice Box* (2011), a version with three different pieces (right)

Summary

We have uncovered all *Pennyhedra* puzzles composed of 24 tetras, thanks to BurrTools [4]. However, additional *Pennyhedron* designs await discovery—because we can always dissect the RD at a finer level.

A nice collection of wood *Pennyhedra* is a set of 10–20 RD, identical in appearance, each composed of two, three, or four pieces and coming apart in different ways. Contact Stephen Chin (chins@ihug.com.au) if you are interested in such a collection. Beware his *6-Finger Pennyhedron*, a “one-piece” RD which does not come apart!

References

[1] Stewart Coffin, Geometric Puzzle Design, CRC Press, 2007.
[2] Stewart Coffin, Ap-Art, A Compendium of Puzzle Designs, August 2003.
[3] Polypuzzles web store, http://www.shapeways.com/shops/polypuzzles
[4] BurrTools, http://burrtools.sourceforge.net/