Pyradox: A Pyramid Packing Paradox

by George Bell

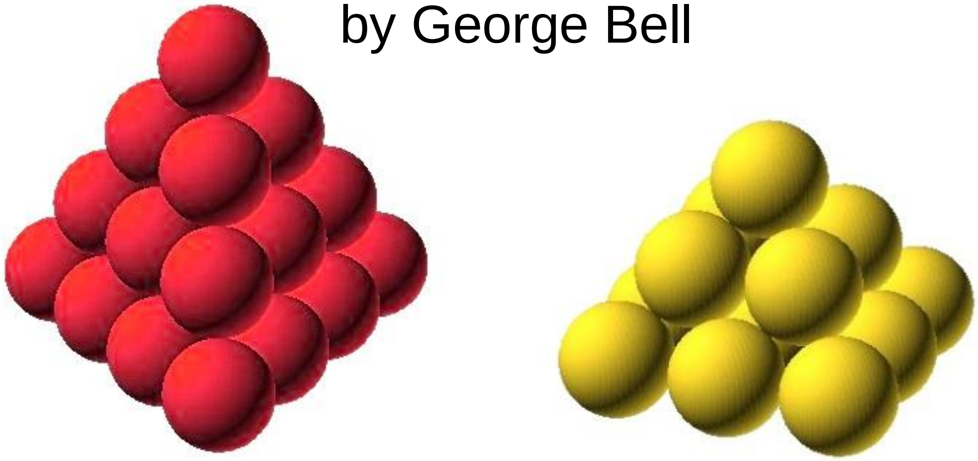

Figure 1. A tetrahedron of height 4 and a 3x3 pyramid of height 3 composed of 20 and 14 balls, respectively.

Puzzle pieces made from identical spheres joined together are called polyspheres [1-3]. They stack naturally into three or four-sided pyramids (Figure 1), like oranges in the grocery store. This leads us to ask: Is there a set of polysphere pieces that can assemble into **both** a three-sided pyramid (tetrahedron) and a four-sided pyramid? This is the idea behind a multiple assembly puzzle I call "*Pyradox*".

Puzzle Geometry

To explain the paradox, I will first prove that such an assembly is impossible, and then show how it can be accomplished! The assembly is impossible due to the basic fact that a tetrahedron and square-base pyramid can never contain the same number of balls. For a tetrahedron of height n, the number of balls is $n(n+1)(n+2)/6$ (tetrahedral numbers) while for the square-base $n \times n$ pyramid the number of balls is $n(n+1)(2n+1)/6$. Table 1 gives these sequences up to n=10.

size (n)	1	2	3	4	5	6	7	8	9	10
Tetrahedron	1	4	10	20	35	56	84	120	165	220
Square-base Pyramid	1	5	14	30	55	91	140	204	285	385

Table 1. The number of balls in tetrahedra and square-based pyramids.

For $n \leq 10$, we see that except for the degenerate case of one ball these sequences never give the same number (although note the near miss of 55 and 56). In 1988 Dutch mathematicians Frits Beukers and Jaap Top proved that there is no number other than 1 common to these two sequences [4].

The resolution of the conundrum is that a "four-sided pyramid" can be built starting from an $n \times m$ **rectangular** base, we call this an "$n \times m$ roof". The name "roof" is appropriate because for $n>m$ the apex is not a single ball, but a line of $n-m+1$ balls. For $n \geq m$ the number of balls in an $n \times m$ roof is $m(m+1)(3n-m+1)/6$, shown in Table 2.

Even among $n \times m$ roofs, tetrahedral numbers are rare. Of particular interest for puzzles is the *4x3 roof* (Figure 2), which has 20 balls, the same as the height 4 tetrahedron. On my computer, I extended this table to *n*=1000000, and found no additional tetrahedral numbers among *n*x(*n*-1) roofs. Also notable because it uses 20 balls is the 7x2 roof. The other tetrahedral numbers in Table 2 are long skinny roofs or those with more than 100 balls, less interesting for puzzles.

	n											
m	2	3	4	5	6	7	8	9	10	11	12	13
2	5	8	11	14	17	**20**	23	26	29	32	**35**	38
3		14	**20**	26	32	38	44	50	**56**	62	68	74
4			30	40	50	60	70	80	90	100	110	**120**
5				55	70	85	100	115	130	145	160	175
6					91	112	133	154	175	196	217	238
7						140	168	196	224	252	280	308
8							204	240	276	312	348	384
9								285	330	375	420	465

Table 2. The number of balls in an nxm roof, tetrahedral numbers in red.

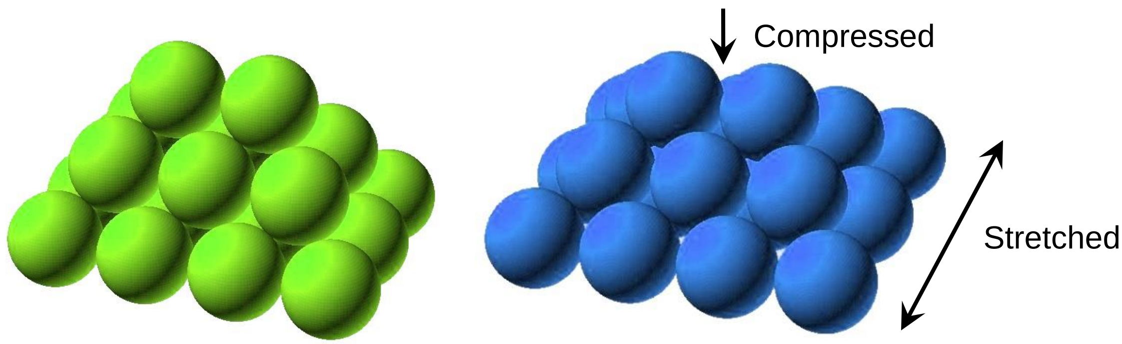

Figure 2. The 4x3 roof and low profile 4x3 roof, both made from 20 balls.

Leonard Gordon discovered a fascinating transformation of any square-base pyramid or roof. Stretch the pyramid in one horizontal dimension by a factor of $\sqrt{2}$ while compressing in the vertical dimension by $1/\sqrt{2}$ (only the centres of the balls move, they remain spheres). The remarkable fact is that the transformed pyramid is still face-centred cubic packing — it is a new kind of pyramid or roof made from the same number of balls. If we perform this transformation on the 4x3 roof, we obtain the “low profile 4x3 roof” (Figure 2). The base of the low profile 4x3 roof has an aspect ratio of 1.045, and visually the base appears square. The first low profile pyramid was a 4x4 used in Leonard Gordon’s multiple assembly puzzle *Warp 30* [5].

Pyradox and other puzzles

Some existing polysphere puzzles for a height 4 tetrahedron can be assembled into other 20-ball shapes. *Kugelpyramide* [6] (1968) is a very early design which can be assembled into 4 different shapes (see Table 3). Others are Piet Hein’s *Pyramystery* [7] (which also assembles into two height 3 tetrahedra) and Leonard Gordon’s *Perplexing Pyramid* [5]. As far as I know, these puzzles were not designed to assemble into so many shapes — they usually come with only one base plate.

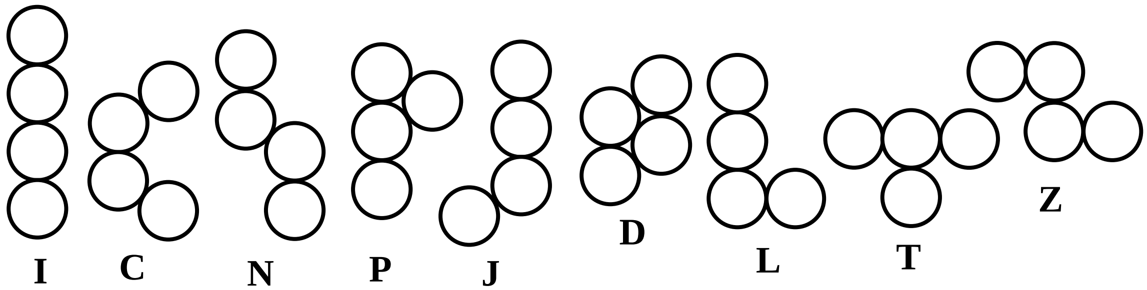

Figure 3. 4-ball polyspheres used in these puzzles, with labels.

My initial design of Pyradox (in 2009) used four non-planar polyspheres with 5-balls each, and assembled into a height-4 tetrahedron or 4x3 roof. Non-planar pieces are very confusing as well as being difficult to manufacture. In my second attempt I used five pieces with 4 balls each. In order to narrow down the number of solutions, I apply the following criteria:

1. Each piece must be planar and composed of 4 balls.
2. All pieces must be different.
3. No linear pieces.
4.

According to BurrTools [8], four designs meet these three criteria and assemble into the height 4 tetrahedron and the 4x3 roof, they are Pyradox 1-4 in Table 2. All four are similar in the way they form the tetrahedron (N and D lie together).

		number of assemblies into			
puzzle	pieces	Tetra4	4x3 Roof	LP 4x3	7x2 Roof
Kugelpyramide	6: 2xI3, D3, C3, D, P	49	123	3	15
Pyramystery	6: I3, 2xD3, C3, D, P	89	157	0	14
Perplexing Pyramid	6: I2, I3, L3, I, L, T	2	3	0	2
Pyradox 1	5: D, J, N, P, Z	2	1	0	1
Pyradox 2	5: C, D, N, P, L	2	2	0	0
Pyradox 3	5: D, J, N, P, L	2	4	0	0
Pyradox 4	5: D, J, N, P, T	2	3	0	1
Triple Pyradox 1	5: D, I, J, P, L	2	3	1	0
Triple Pyradox 2	5: 2xJ, 2xP, T	1	1	1	0

Table 3. Puzzles which can form the height 4 tetrahedron, 4x3 roof, low profile 4x3 roof and 7x2 roof (for pieces not defined in Figure 3, see [3]).

If we remove criterion 3, we find a **unique** puzzle which assembles into the first three 20-ball pyramids in Table 3. The *Triple Pyradox*! If we instead remove criterion 2, we find another unique puzzle with exactly 1 assembly into each pyramid.

We have given a number of polysphere puzzles which can assemble into tetrahedra and four-sided pyramids. Each assembly requires a base plate or the pyramid will simply fall apart. If you own one of the puzzles in Table 3 and want to make more base plates, here are some tips. For balls of diameter *d* and a base of thickness *t*, use hole centres separated by *d*, except for the low profile pyramid which in one direction is $\sqrt{2}d$. For $t \le d/2$, the holes should have a maximum radius of $\sqrt{t(d-t)}$. The hole centres should be positioned very accurately, a laser-cut base is ideal.

References

[1] George Bell, Classification of Polyspheres, CFF81
[2] George Bell, Sphere Octahedron Puzzles, CFF84
[3] George Bell, Solving J. Gordon's Giant Pyramid, CFF 90
[4] Frits Beukers and Jaap Top, "On Oranges and Integral Points on Certain Plane Cubic Curves." Nieuw Archief voor Wiskunde 6, 203-210, 1988
[5] Kadon Enterprises, http://www.gamepuzzles.com/
[6] Bernhard Wiezorke, Compendium of Polysphere Puzzles, 1996
[7] sold as *Cannonball Pyramid* at http://www.creativecrafthouse.com/
[8] BurrTools, http://burrtools.sourceforge.net/