Hexagonal Ball Pyramid Puzzles

by George Bell

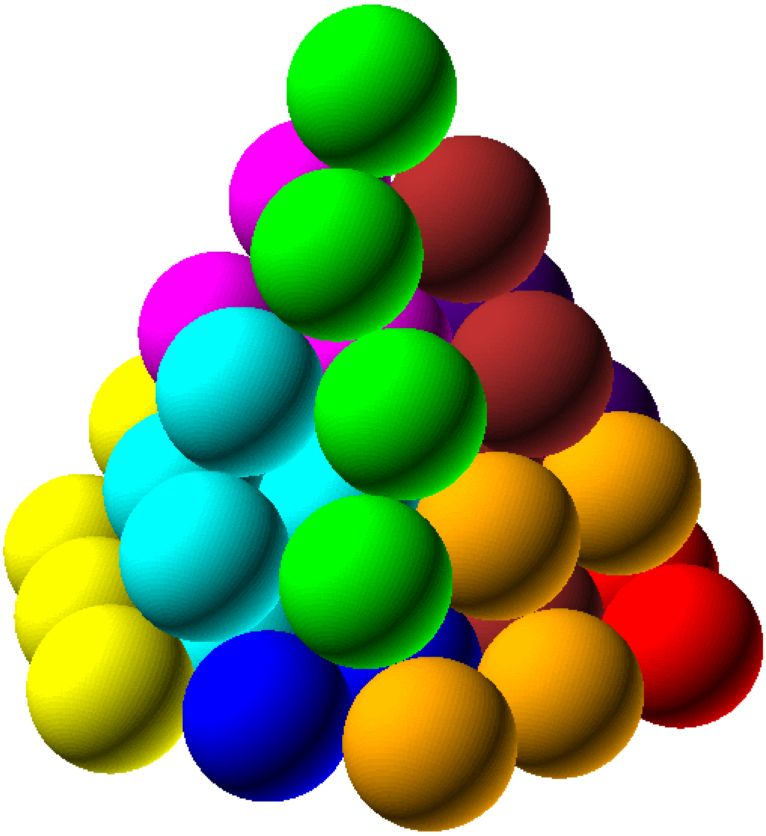

Figure 1. A 42-ball hexagonal pyramid (Zig-Zag pyramid puzzle).

Figure 1 shows a ball puzzle where the target shape is a hexagonal ball pyramid. A hexagonal ball pyramid is formed by stacking spheres in layers, where every other layer is a hexagon (Figure 2). The number of balls in a hexagonal pyramid is given by the sequence A019298 [1]: 1, 4, 11, 23, 42, 69, 106, 154, 215, … A general formula for the number of balls in a hexagonal pyramid of n layers is $\lfloor (n^2+1)(2n+3)/8 \rfloor$. Here the brackets represent the floor function, rounding down to the nearest integer.

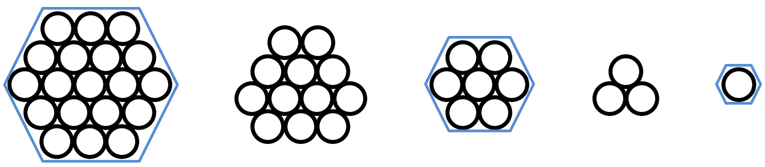

Figure 2. The 5 layers in a 42-ball hexagonal pyramid.

Although it appears similar to other ball puzzles, a hexagonal ball pyramid is actually very different. While virtually all ball puzzles are based on face-centred cubic sphere packing, a hexagonal ball pyramid is based on hexagonal close packing. This article will show how puzzle pieces based on hexagonal close packing differ from the usual polyspheres [2] and how to solve these puzzles using BurrTools [3].

Sphere packing and puzzle pieces

In Figure 3 we pack spheres in a hexagonal layer (A). The second layer (B) is packed in the indentations of the first. At the third layer, we have two options: we can either repeat pattern A or we use the third set of indentations (C). There are an infinite number of layering patterns taken from A, B, C, but if we alternate ABAB… we obtain hexagonal close packing, while the pattern ABCABC… gives face-centred cubic. Both these packings have the same density, now known to be the greatest possible [4].

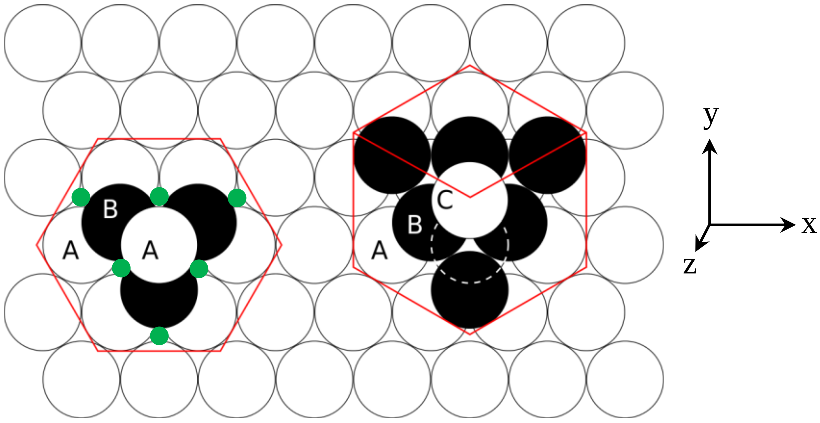

Figure 3. Sphere packing options (from [5]).

We note in Figure 3 that the left shape outlined in red is an 11-ball hexagonal pyramid, while on the right is "the cube" from which face-centred cubic is named (13 spheres). The green dots mark the location of holes (where the C spheres would go) which tunnel through the entire hexagonal close packing lattice. There are no such tunnels in face-centred cubic. If you are tasked with efficiently packing an infinite number of spheres of radius 1 together with an infinite number of rods of diameter 0.309 or less, you are in luck! Hexagonal close packing is the only way to accomplish it.

If you desire real-world (nano) examples of these packings, look no further than your own jewelry. Copper, aluminium, silver and gold have atoms arranged in face-centred cubic packing, while cobalt, cadmium, zinc and titanium prefer hexagonal close packing [6].

For face-centred cubic packing, there is a simple way to represent the sphere centres. Take all integer coordinates (x, y, z) such that the sum of the coordinates x+y+z is even and use spheres of diameter $\sqrt{2}$. We can generate these points by taking all integer linear combinations of (1, -1, 0), (0, 1, -1) and (0, 0, 2), resulting in the sphere centre equation

$$(i, j - i, 2k - j) \tag{1}$$

where *i, j* and *k* are arbitrary integers. It is important to realize that the coordinate system in Equation (1) is **not** the same as that shown in Figure 3, there is a rotation involved in going between the two. BurrTools [3] uses the coordinate system in Equation (1), whereas when building the pyramid, layer by layer, we are in the coordinate system of Figure 3. Each layer in Figure 3 corresponds to a different value of the coordinate sum x+y+z in Equation (1).

In hexagonal close packing, there are only two types of layers, which seems simpler. But this is deceiving, in fact things are more complicated (this will become a common theme). Compared to Equation (1), we need to shift the spheres in the 3rd and 4th

layers by (1/3, 1/3, -2/3), so the sphere centres will no longer have integer coordinates. We can keep them at integer coordinates by expanding the lattice by a factor of 3 in all dimensions; we then find that a general formula for the sphere centres is

$$(3i + \lfloor k/2 \rfloor, 3j - 3i + \lfloor k/2 \rfloor, 6k - 3j - 2\lfloor k/2 \rfloor) \qquad (2)$$

where *i, j* and *k* are arbitrary integers, and we use spheres of diameter $3\sqrt{2}$.

Polyspheres are puzzle pieces obtained by joining spheres in the face-centred cubic lattice [2]. For any polysphere, the angle between three balls which touch can only be 60°, 90°, 120° or 180°. This gives us exactly four 3-polyspheres (made from 3 balls), shown in Figure 4, together with their descriptive names.

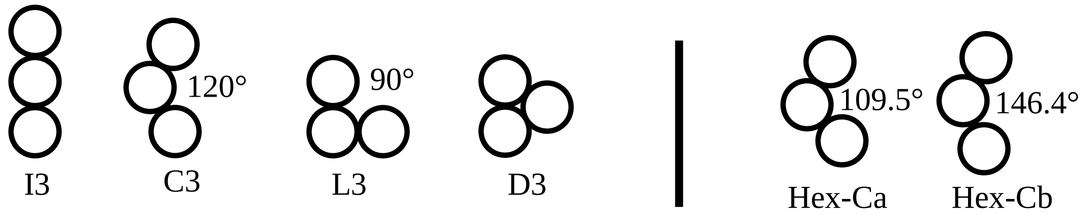

Figure 4. The four possible 3-polyspheres (left), two new pieces in hexagonal close packing (right).

All planar polyspheres (by planar we mean that sphere centres lie in the same plane) satisfy certain angle restrictions. Planar polyspheres fall into two categories:

1. Pieces that use only 60°, 120° and 180° and are equivalent to polyhexes.
2. Pieces that use only 90° and 180° and are equivalent to polyominoes.

For more information about polyspheres and these categories see [2].

In hexagonal close packing, things are more complicated (surprised?). We find that in addition to the four angles above there are two new angles: $\cos^{-1}(-1/3) \cong 109.5°$ and $\cos^{-1}(-5/6) \cong 146.4°$. We call pieces created from hexagonal close packing of spheres **hexaspheres**. There are six possible 3-hexaspheres, all shown in Figure 4 (I apologise in advance for the awkward naming system).

These two additional angles make hexasphere puzzles more complex than polysphere puzzles. In particular, it is very hard to distinguish the angles of 109.5° and 120°. Puzzle pieces must be made with great accuracy [7]. I have not attempted the difficult task of counting all *n*-hexaspheres beyond *n* = 3.

There are also many polysphere pieces which are not hexasphere pieces. In regard to planar pieces, we have the following:

1. Any polyhex piece is also a hexasphere (no restriction).
2. A polyomino piece must fit in a 2 x *k* tray to be a hexasphere (*k* is arbitrary).
3. Any planar piece with an angle of 109.5° can **only** have angles of 109.5°.
4. Any planar piece with an angle of 146.4° can **only** have angles of 146.4°.

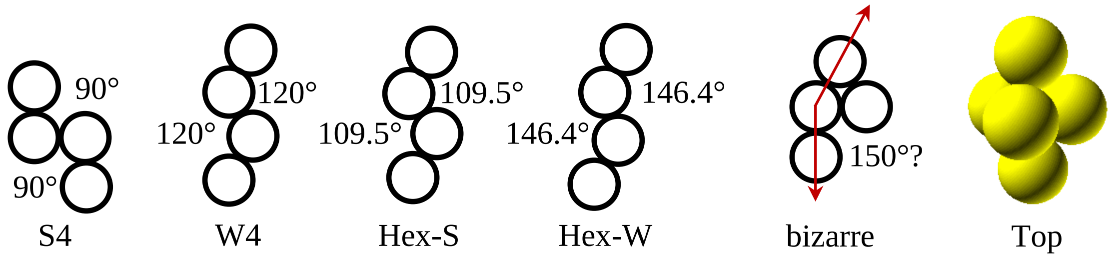

Figure 5. The four Zig-zag pieces (planar), a bizarre piece, and Top.

These last two indicate that planar pieces made using the two new angles are very restricted. This gives rise to what I call zig-zag pieces, of which there are four baffling types (Figure 5). Pieces using only angles of 109.5° can only be oriented vertically (in Figure 1), while pieces using only angles of 146.4° can only be aligned parallel to the edges of a hexagonal pyramid (green piece in Figure 1).

3D (non-planar) hexaspheres can have very bizarre shapes. For a 3D piece, it is possible to have any or all of the six possible angles. The “bizarre” piece in Figure 5 can be seen on the outside of Hexagonal ball pyramids and appears to use an angle of 150°, not on our list of possible angles! Upon closer examination, we find that this hexasphere is not planar, but is slightly bent out of the page, the actual angle is 146.4°.

A new symmetrical 3D piece is “Top” (named by Leonard Gordon [8]) shown in Figure 5. This 5-ball piece is made from D3 by gluing a ball to the top, plus one on the bottom, it is not a polysphere. This piece is so called because it makes a nice spinning top. “Top” can only be placed vertically in the orientation of Figure 1 (*z*-axis in Figure 3).

Solving hexagonal pyramid puzzles

BurrTools [3] has a “Spheres” space grid, but this builds polyspheres in the face-centred cubic lattice. For many years, I assumed that hexagonal ball pyramid puzzles could not be solved by BurrTools.

Several recent CFF articles have shown how BurrTools can solve many problems which it was not designed for. It seemed likely that there existed some trick to modeling hexagonal close packing in BurrTools, but how? After deriving Equation (2), I suddenly realized that it was a subset of the full face-centred cubic lattice of Equation (1). Therefore, it is possible to model hexagonal close packing puzzles in BurrTools by using only part of the “Spheres” space grid.

In order to model a hexagonal pyramid, we carefully select spheres only at the coordinates specified by Equation (2). Table 1 gives sphere coordinates for a 42-ball hexagonal pyramid. Most of the space grid is left empty, and because the spheres displayed are too small by a factor of 3 everything appears disconnected.

layer	Sphere centre coordinates
Base x+y+z=0	[-6,0,6],[-3,-3,6],[0,-6,6] [-6,3,3],[-3,0,3],[0,-3,3],[3,-6,3] [-6,6,0],[-3,3,0],[0,0,0],[3,-3,0],[6,-6,0] [-3,6,-3],[0,3,-3],[3,0,-3],[6,-3,-3] [0,6,-6],[3,3,-6],[6,0,-6]
2 x+y+z=6	[-3,3,6],[0,0,6],[3,-3,6] [-3,6,3],[0,3,3],[3,0,3],[6,-3,3] [0,6,0],[3,3,0],[6,0,0] [3,6,-3],[6,3,-3]
3 x+y+z=12	[1,4,7],[4,1,7] [1,7,4],[4,4,4],[7,1,4] [4,7,1],[7,4,1]
4 x+y+z=18	[4,7,7],[7,4,7] [7,7,4]
top (=24)	[8,8,8]

Table 1. Sphere centres for a 42-ball hexagonal pyramid

For the reasons given in the next paragraph, all pieces should also be defined on the sphere coordinates in Table 1. One way to ensure this is to copy one size of hexagonal pyramid, then remove all the spheres not in the piece you are creating. The supplemental material for this CFF article includes BurrTools files with hexagonal ball pyramids up to 106 balls, plus all the puzzles in this article [10].

A warning in the BurrTools user guide [3] cautions against using disconnected pieces in the Spheres grid. The problem is that an infinite number of potential rotations may need to be considered as disconnected pieces grow. I believe that sticking with the sphere coordinates in Table 1 ensures that the set of symmetry transformations included in BurrTools are sufficient to solve these puzzles.

Hexagonal ball pyramid puzzles

The first person to create hexagonal ball pyramid puzzles was Leonard Gordon [8]. His intimidating **Teepee** (1984) forms a 7-layer, 106-ball hexagonal pyramid. This puzzle is very rare—I know of only one copy, owned by Stan Isaacs (Figure 6). In the physical puzzle, all pieces of the same type are not the same colour.

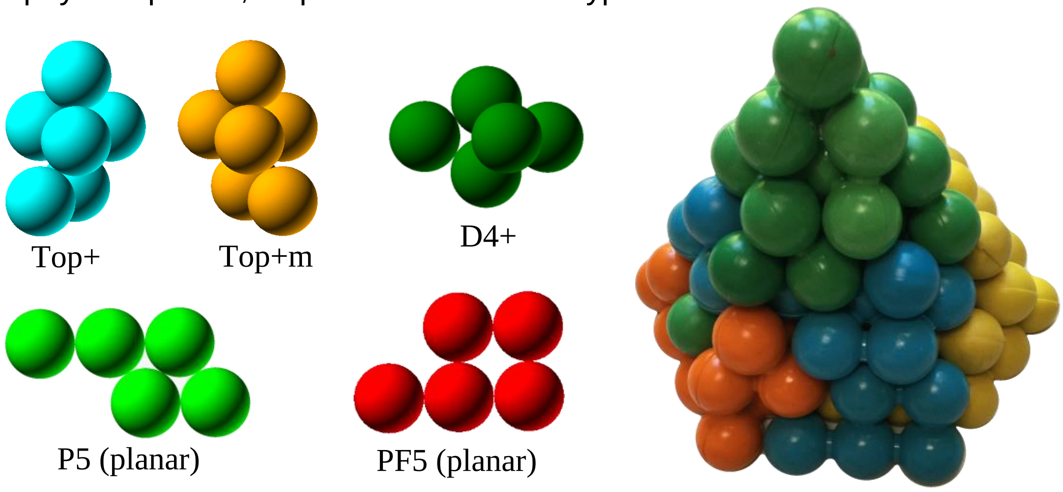

Figure 6. The Teepee puzzle by Leonard Gordon (1984), photo by Stan Isaacs.

Leonard Gordon mentions **Teepee** in [8], but it is not fully described. Using Stan's copy we have determined that the puzzle has 20 pieces of 5 types, and 11 pieces are 3D. The piece I call **D4+** can be described as adding one ball to **D4** (it is also a polysphere). **Top+** can be obtained by adding one ball to **Top**, **Top+m** is its mirror image.

According to Leonard Gordon, "(Teepee) appears formidable, but is not difficult if you study hexagonal close packing" [8]. BurrTools solves the **Teepee** puzzle in about 20 minutes on my PC, giving 4005 solutions. Stan Isaacs described to me a symmetrical solution, easier to remember and likely what the designer intended. After adding colour restrictions in BurrTools, I found these symmetrical solutions (included with the supplemental material). Without Stan's copy of **Teepee** and his memory of the solution, the details behind this puzzle might have been lost forever.

In Leonard Gordon's Notes [8], he lists three other puzzles which form a 42-ball hexagonal pyramid. Table 2 lists these puzzles, together with the number of solutions reported by BurrTools. For pieces not defined in this document, see [8], [9] or the BurrTools supplementary material [10].

puzzle	total balls	pieces	no. of solns
Teepee	106	20: 3xTop+, 3xTop+m, 5xD4+, 6xP5, 3xPF5	4005
Gordon puzzle #1	42	11: 3xHex-S, 4xC4, 2xW4, D4, I2	2
Gordon puzzle #2	42	11: 2xHex-S, 5xP4, 2xD4, 2xL3	41
Gordon puzzle #3	42	11: Top, 2xHex-S, S4, W4, L4, C4, D4, J4, L3, I2	7
Zig-Zag Pyramid	42	10: 2xTop, Hex-S, Hex-W, S4, W4, L4, J4, D4, P4	1
Mini Zag-Zag	23	7: Hex-W, S4, W4, Hex-Ca, C3, L3, I2	2
Seven L Pyramid	23	8: 7xL3, I2	2
Six L Pyramid	23	7: Hex-S, L4, 5xL3	1
Six L Pyr. (alt)	23	7: 3xL3, 3xL2, I2	8
Fourteen L Pyr.	42	14: Hex-Ca, 13xL3	42

Table 2. Hexagonal ball pyramid puzzles.

I designed the **Zig-Zag Pyramid** with the goal of using all four zig-zag pieces. A 23-ball hexagonal pyramid cannot include both Hex-S and Hex-W (because they must both use the apex), so the smallest hexagonal pyramid which can use all four has 42 balls. Table 1 includes a 10-piece design with a unique solution. I have also included **Mini Zig-Zag Pyramid**, a 23-ball design which uses three of the four zig-zag pieces.

Summary

Most ball puzzles are based on face-centred cubic packing, which gives polysphere puzzle pieces. Hexagonal ball pyramids are very different, because they are based on hexagonal close packing. Hexaspheres are puzzle pieces based on hexagonal close packing of spheres. They are related to polyspheres but are more complex. Instead of four possible angles between touching balls, there are six. Even small hexagonal ball pyramid puzzles can be baffling, we have showed how they can be solved using BurrTools [3].

References

[1] The On-line Encyclopedia of Integer Sequences (OEIS), https://oeis.org/
[2] George Bell, Classification of Polyspheres, CFF 81, March 2010.
[3] BurrTools, http://burrtools.sourceforge.net/
[4] Thomas Hales, “A Formal Proof of the Kepler Conjecture”, Forum of Mathematics, Pi. 5: e2, May 2017, *doi:10.1017/fmp.2017.1* .
[5] By en:User:Twisp - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=3501327
[6] Nevill Gonzales Swacki and Teresa Swacka, Basic elements of Crystallography, Pan Standford, 2010
[7] George Bell, Making your own ball puzzles, CFF 95, November 2014.
[8] Leonard Gordon, Some Notes on Ball-Pyramid and Related Puzzles, 1986. For a downloadable pdf, see http://www.gibell.net/puzzles/
[9] George Bell, Pyradox: A Pyramid Packing Paradox, CFF 94, July 2014.
[10] CFF Website, http://CFF.Helm.lu