More Balls in a Box

by George Bell

Introduction: A brief history of ball packing puzzles

In a 1960 Scientific American column, Martin Gardner asked for the maximum number of 1” diameter spheres which can be packed into a box with inside dimensions 10” x 10” x 5” [1]. Gardner warned readers that this is “an exceedingly tricky little problem”, and with more than 500 pieces it is too complex to make a good physical puzzle.

Figure 1. *Warp-30* by Leonard Gordon (1988), photo courtesy Kate Jones.

To create puzzles of reasonable size people started with many fewer spheres and joined them together to form pieces, called polyspheres.

One of the first polysphere box-packing puzzles was *Warp-30* by Leonard Gordon [2], sold by Kadon Enterprises [3] in 1988 (Figure 1). This puzzle involves packing eight pieces composed of 30 balls into two boxes as well as building two pyramids (in this document I will use the words “sphere” and “ball” interchangeably). Another Leonard Gordon design from this time period is *Ell of a Puzzle* [2] (8 pieces, 32 balls).

In the 1990’s Wolfgang Schneider became interested in this type of puzzle and published “Balls in a Box” in CFF38 [4]. Two of his designs are *Balls in a Box* (1995) and *Cube Ball Ogy* (1995) [5], made by his company kubi-games. Sadly, Wolfgang Schneider passed away in 2003.

Perhaps the most well-known puzzle of this type is Stewart Coffin’s *Ball Room* (design #197-A) [6], Jerry Slocum’s IPP25 exchange puzzle (Helsinki, 2005). The four pieces are composed of 14 balls, and pack into the box (Figure 2) as well as building two pyramids. Iwahiro (Hirokazu Iwasawa) also has one puzzle which fits into this category: *Dango Box* (9 pieces, 30 balls) made by himself and later by Philos.

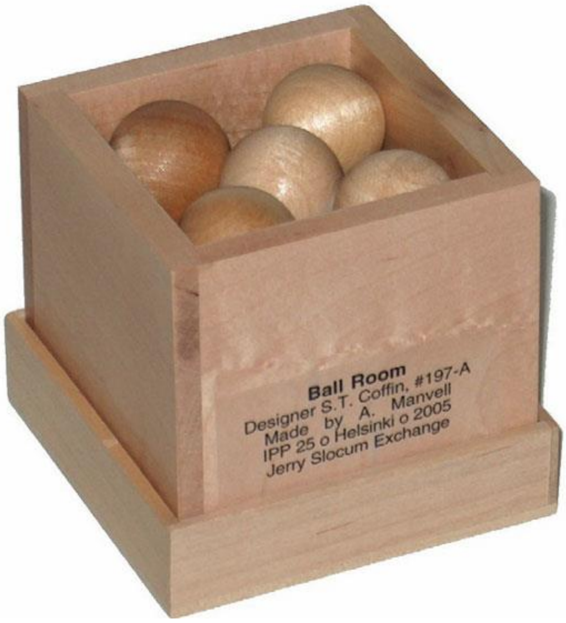

Figure 2. *Ball Room* by Stewart Coffin (2005), photo courtesy Nick Baxter.

Packing spheres in square layers

Most of these puzzles place spheres diagonally in square packing layers as shown in Figure 3. We show only the first two layers—after this the layering pattern is repeated, which results in face-centred cubic (FCC) sphere packing. Conveniently, this diagonal packing is exactly the way sphere puzzles are entered into BurrTools [7]. Another nice feature of this packing scheme is that all square packing planes are parallel to the faces of the box, this means that every box face looks like Figure 3. Hexagonal packing planes are always present, they are parallel to the faces of an internal tetrahedron (Figure 3).

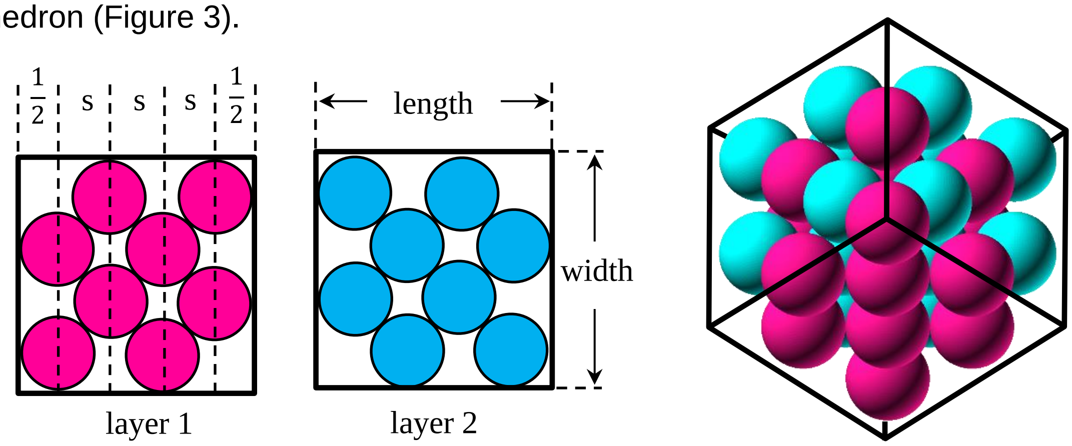

Figure 3. Packing diagonally in a square grid with *a*=*b*=*c*=*4*, the 3D packing tilted to show hexagonal packing planes.

We should mention that there are other ways to pack spheres in boxes even using square packing layers. Part of the appeal of *Warp-30* and the Wolfgang Schneider puzzles is the different ways the pieces can be packed into boxes. See [4] for details.

If the BurrTools spacegrid is *a* x *b* x *c*, the number of balls which can be packed in the box using this packing scheme is $\lceil abc/2 \rceil$. Here the brackets represent the ceiling function, rounding up to the nearest integer. The puzzles in the introduction use the following box sizes: *Warp-30,* 5 x 4 x 3 (30 balls); *Ell of a Puzzle*, the Schneider puzzles and *Dango Box* 4 x 4 x 4 (32 balls); *Ball Room* 3 x 3 x 3 (14 balls).

If *a* or *b* is even then each horizontal layer contains $h = ab/2$ balls. If *a* and *b* are both odd then the horizontal layers alternate between $h = \lceil ab/2 \rceil$ and $h - 1$ balls (*Ball Room* is an example of this, the horizontal layers alternate between 5 and 4 balls).

What are the inside box dimensions? Suppose we use balls of diameter 1. The distance between square packing planes (for face-centred cubic) is $s = \sqrt{2}/2$, so the box length is $1 + (a - 1)s$ (as shown in Figure 3 for *a* = *4*). The same holds for the other dimensions *b* and *c*. For Martin Gardner's problem [1], the largest spacegrid which fits in the box is 13 x 13 x 6 with 507 balls. This is not an efficient packing due to large gaps along the edges of the box, but it is better than cubic packing (500 balls).

Wood balls are available only in certain sizes. Woodworkers can calculate the box size required and create a custom box. If the pieces are 3D printed you can start with a commercially made box and adjust the ball size to fit the box. If you use wood balls and commercially made boxes, the reality is that the sizing rarely works out.

An interesting modification of this type of puzzle is to replace the spheres with rhombic dodecahedra. We can also truncate the rhombic dodecahedron which is equivalent to an edge-beveled cube. Figure 4 shows the transformation of a rhombic dodecahedron into a cube by truncation of the degree 4 vertices. When edge-beveled cubes are substituted for spheres, usually the box must be resized to fit them. Only in one special case is the box size unchanged—when the distance between opposing hexagonal faces is the same as that between opposing square faces (the middle red solid in Figure 4). This sizing trick works only for the boxes in this section.

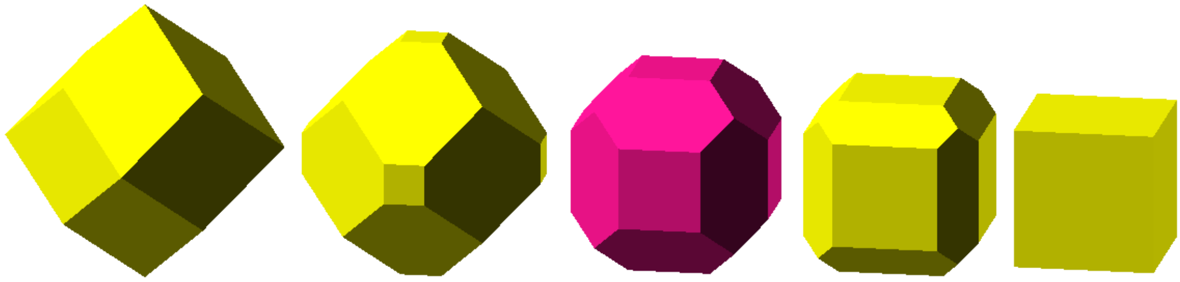

Figure 4. Truncation of a rhombic dodecahedron to a cube.
The middle version is special because it requires no change to the box size.

When the pieces are made from edge-beveled cubes, the puzzle may become interlocking. It may be necessary to assemble the pieces outside the box and then slide the assembly in. One may not even need the box!

Packing spheres in hexagonal layers

Why not pack the spheres in hexagonal layers? Figure 5 shows one option. From a puzzle design standpoint, there are several problems. First, the box isn't a cube—in general all three dimensions will be different (which means a custom box). Second, we can't have hexagonal (or square) packing on the vertical box faces. The result is that some layers must have annoying gaps, in Figure 5 all three layers have gaps.

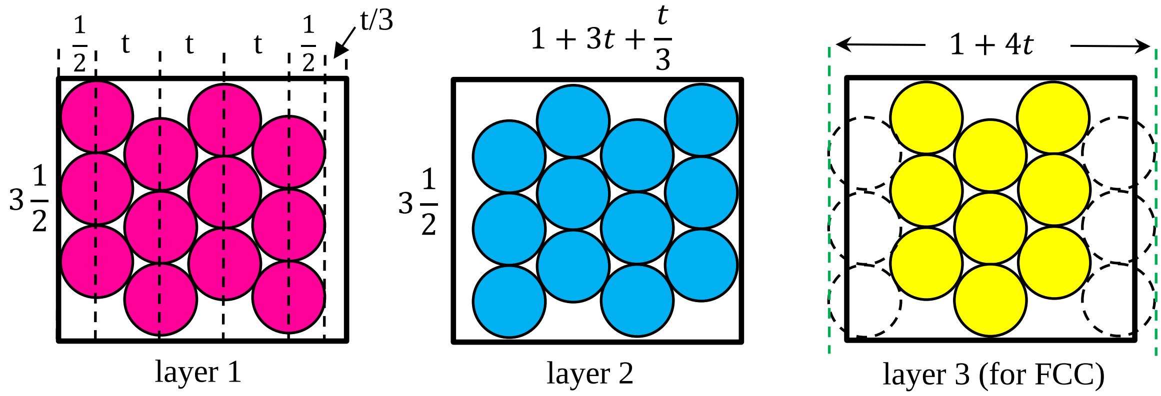

Figure 5. Spheres packed in hexagonal layers, a=4, b=3.

There is another problem. If alternate layers 1 and 2, this does not give face-centred cubic packing! As explained in [8], we get hexagonal close packing. While it may be possible to pack the box using polyspheres, in general we may need the more complicated hexaspheres [8]. When there are more than 2 layers, solving such puzzles in Burrtools is possible, but is more difficult [8].

In order to preserve face-centred cubic packing, it is necessary to add layer 3, shown in Figure 5. This 9-ball pattern does not fill the box very efficiently, if we make the box wider, the six balls marked by the dashed lines will fit, but there will be larger gaps in layers 1 and 2. I think the best option is to alternate layers 1 and 2, then each layer contains 12 balls. Note that when the packing is only two layers high, there are insufficient layers to distinguish face-centred cubic from hexagonal close packing.

In Figure 5, $t = \sqrt{3}/2$ is the distance between hexagonal packing lines. In general, the box length is $1 + (a - 1)t + t/3$ and the box width is $b + 1/2$. The box height is $1 + (c - 1)u$ where $u = \sqrt{6}/3$ is the distance between hexagonal packing planes (same as the height of a regular tetrahedron of edge length 1). The total number of balls in the box is $\lceil abc \rceil$ (rounding up is still necessary because *b* can be a half integer).

This packing is very efficient for Martin Gardner's problem [1] (oriented the optimal way). We use *a* = 11 columns of balls alternating between 4 and 5 balls (*b* = 4.5), and *c* = 12 layers for a total of 594 balls (this is the maximum possible [1]).

Another option is to use a box which is a regular hexagon, shown in Figure 6. The length of the hexagonal box is $1 + 3t + t/3$ (same as the Figure 5 box). Alternatively, we can use a cylindrical box of inside diameter $1 + \left(4\sqrt{7}\right) t/3$.

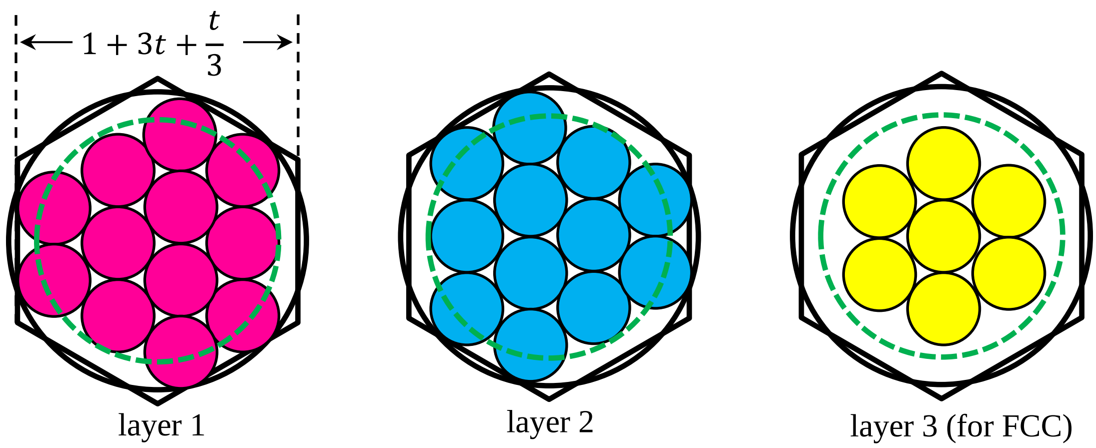

Figure 6. Spheres packed in a hexagonal or cylindrical box.

As with the box of Figure 5, if we alternate layers 1 and 2 we have hexagonal close packing. In order to get face-centred cubic we need to add layer 3 with only 7 balls and lots of empty space. Another way to preserve FCC is to shrink the box as indicated by the dashed green circle (diameter $1 + 8\,t/3$) which encloses 6, 6, and 7 balls per layer. However, in the next section, we'll choose not to preserve FCC.

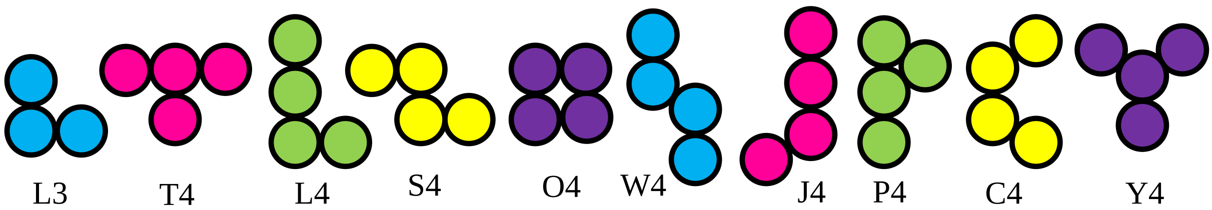

Figure 7. Names for 10 planar polysphere pieces (left 5 are orthogonal).

New Puzzles

The boxes in Figures 5 and 6 both have 12 balls per layer (if we alternate layers 1 and 2). The simplest boxes are two layers tall and fit 24 balls. We will call the box in Figure 5 a "2-layer rectangular box" and the round box in Figure 6 a "2-layer cylindrical box".

Planar polyspheres with 90 degree angles (orthogonal polyspheres) are interesting to use in these boxes because they cannot lie flat in a layer. Eight copies of L3 can fill the 2-layer rectangular box, and six copies of T4 can fill the 2-layer cylindrical box. These are the only solutions using identical copies of an orthogonal polysphere.

Figure 8. *Chocolate Box* by George Bell, box made by Stephen Chin, photo courtesy Nick Baxter. Right: coordinate motion assembly.

Packing six T4 into the cylindrical box is not difficult, but the pieces cannot be added one at a time because there is not enough room to insert the last piece. The easiest way to assemble this puzzle is outside the box, then lowering the assembly into the box. Made from edge-beveled cubes, the pieces become interlocking and can be assembled using coordinate motion. I call this puzzle *Chocolate Box* (Figure 8). Unfortunately, coordinate motion is not required to assemble this puzzle, it is easier to assemble by slotting together identical halves.

layers	pieces	no. of solns rect.	cyl.
2	8: 8xL3	79	0
2	6: 6xT4 (*Chocolate Box*)	0	1
2	6: T4, L4, S4, W4, J4, P4	5	1
2	6: T4, L4, S4, O4, C4, P4	1	0
3	12: 12xL3	0	1
4	16: 16xL3	12,007	30
3	9: L4, T4, O4, W4, C4, D4, J4, Y4, P4	4	1

Table 1. Puzzles using the rectangular box of Figure 5 or the cylindrical box of Figure 6.

Table 1 summarizes puzzles using these rectangular and cylindrical boxes with 2 to 4 layers. When more than two layers are used, we alternate layers 1 and 2. For the puzzles with 3 or 4 layers, the packing is not face-centred cubic. Solving these puzzles in BurrTools [7] requires special tricks as discussed in [8]. In addition, edge-beveled cubes cannot be substituted for spheres.

Summary

We began this journey by considering ways that spheres can pack into boxes. Along the way, the boxes morphed into cylinders, and the spheres morphed into rhombic dodecahedra! *Chocolate Box* (Figure 8) uses pieces made from (edge-beveled) cubes which must be packed into a cylindrical box. It is not easily identifiable as a ball packing puzzle, but is still based on the fascinating geometry of sphere packing.

References

[1] Martin Gardner, New Mathematical Diversions, MAA 1995, p. 86 and 90.
[2] Leonard Gordon, Some Notes on Ball-Pyramid and Related Puzzles, 1986. For a downloadable pdf, see http://www.gibell.net/puzzles/
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[6] Stewart Coffin, AP-ART — A Compendium of Geometric Puzzles, 2018.
[7] BurrTools, http://burrtools.sourceforge.net/
[8] George Bell, Hexagonal Ball Pyramid Puzzles, CFF106, July 2018, pp. 24–29.