Lominoes for G4G9

George Bell
gibell@comcast.net

Lominoes are L-shaped polyominoes of width one. The lomino "$Li \times j$" can be obtained by taking a rectangle with i columns and j rows, and removing all but the bottom row and leftmost column. Because we consider "free" lominoes (which can be flipped over), $Li \times j$ is the same as $Lj \times i$, so we use $i \geq j$. Lominoes were named by Alan Schoen [1], he considers a "standard set of lominoes of order 8" to be all $Li \times j$ with $i \leq 8$ and $j \leq 8$. In contrast, I consider here lominoes sorted by area.

The smallest lomino, $L2 \times 2$, has area 3, and in general the area of $Li \times j$ is $i + j - 1$. $L2 \times 2$ is the only lomino with area 3, and $L3 \times 2$ is the only lomino with area 4. However, there are two lominoes with area 5: $L3 \times 3$ and $L4 \times 2$. In general, the number of lominoes of area a is $\lfloor (a-1)/2 \rfloor$. Figure 1 shows the three lominoes of area 7 (we shall resist calling them "heptlominoes", for obvious reasons). Table 1 shows a count of lominoes by area.

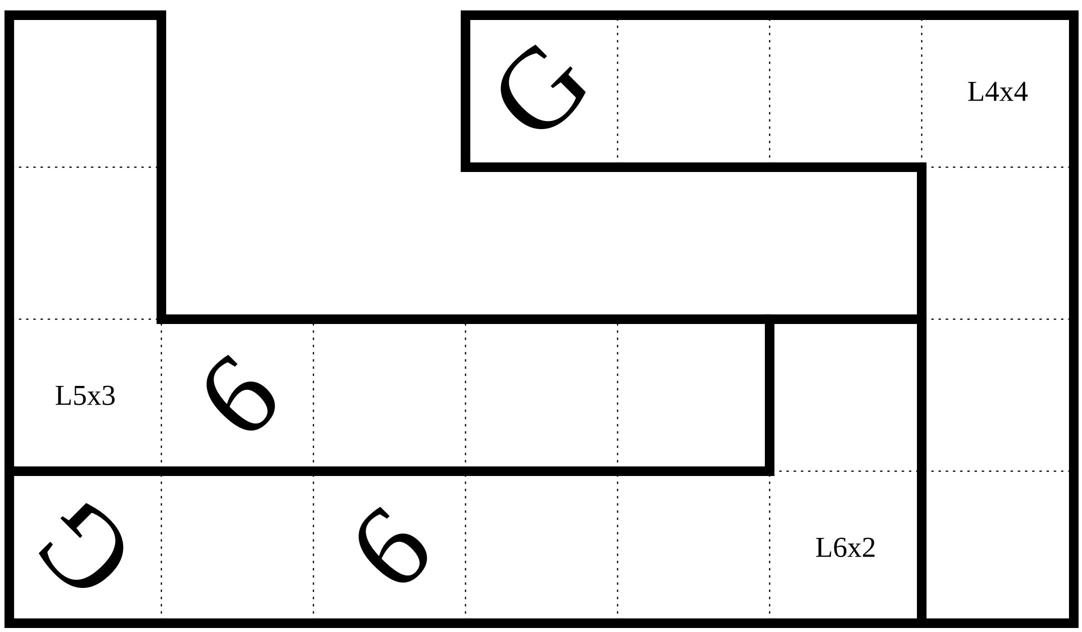

Figure 1: The three lominoes of area 7: $L6 \times 2$, $L5 \times 3$ and $L4 \times 4$.

Suppose we take all lominoes with area less than or equal to A. Can these potentially form a square? From Table 1, we see that the cumulative area never hits a perfect square for $A \leq 11$, and using a computer one finds no perfect square for $A \leq 100,000$.

Suppose instead that we consider only lominoes of area 7, 8, or 9. From Table 1 we see that the total area of these 10 pieces is $21 + 24 + 26 = 81$. Can these pieces be packed into a 9×9 square? Indeed they can.

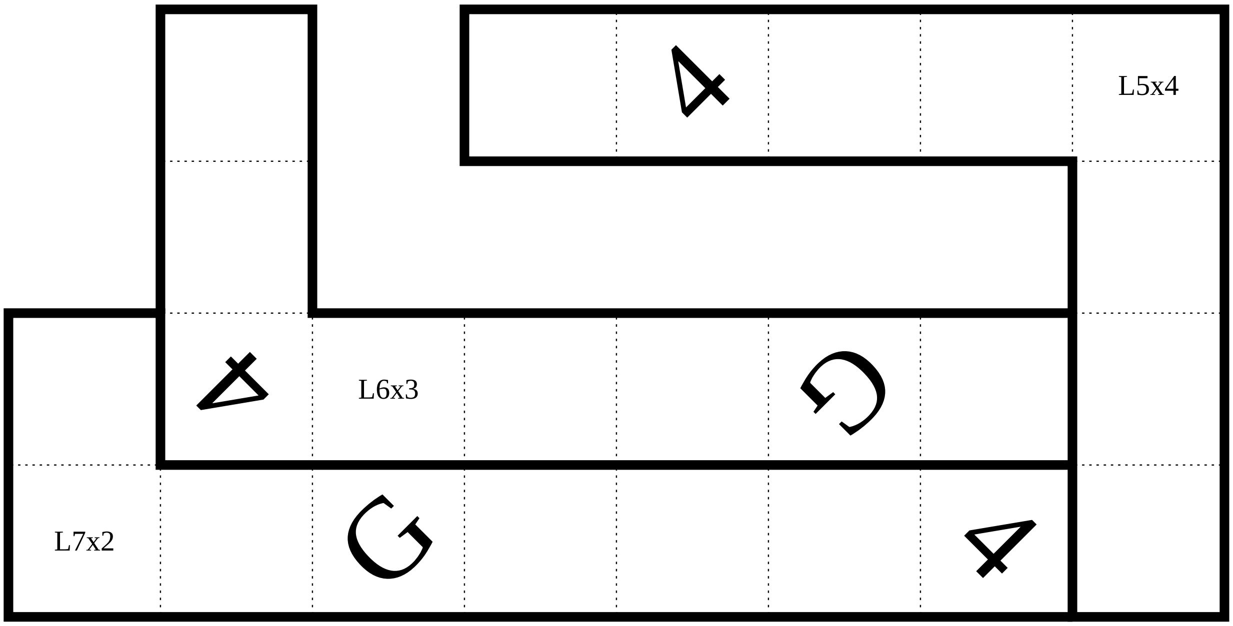

Figure 2: The three lominoes of area 8: $L7 \times 2$, $L6 \times 3$ and $L5 \times 4$.

A Puzzle for G4G9: Lomino81

Figures 1,2 and 3 contain all lominoes of area 7, 8, and 9, respectively. Cut these pieces out along the solid lines (to avoid cutting up this book, you can download a pdf copy of this document at [3]). Your challenge is to pack the ten pieces into a 9×9 square. There are 9 solutions to this puzzle, not counting rotations and reflections. However, there is only one solution where all the pieces are face up—the lettering provides a clue to help you find this solution. No solution is given here, you can find a solution in [3].

After you solve this puzzle, you can use a subset of these pieces to form a 8×8 square, and 14 rectangles of various sizes as indicated in Table 2. In order to solve these puzzles,

Area	no. of lominoes	Total area	Cumulative area
3	1	3	3
4	1	4	7
5	2	10	17
6	2	12	29
7	3	21	50
8	3	24	74
9	4	36	110
10	4	40	150
11	5	55	205

Table 1: Number of lominoes by area.

you must decide which piece (or pieces) to leave out, and pack the remaining pieces into a rectangle of the required size. In order to solve most of these problems, you will need to flip some of the lominoes.

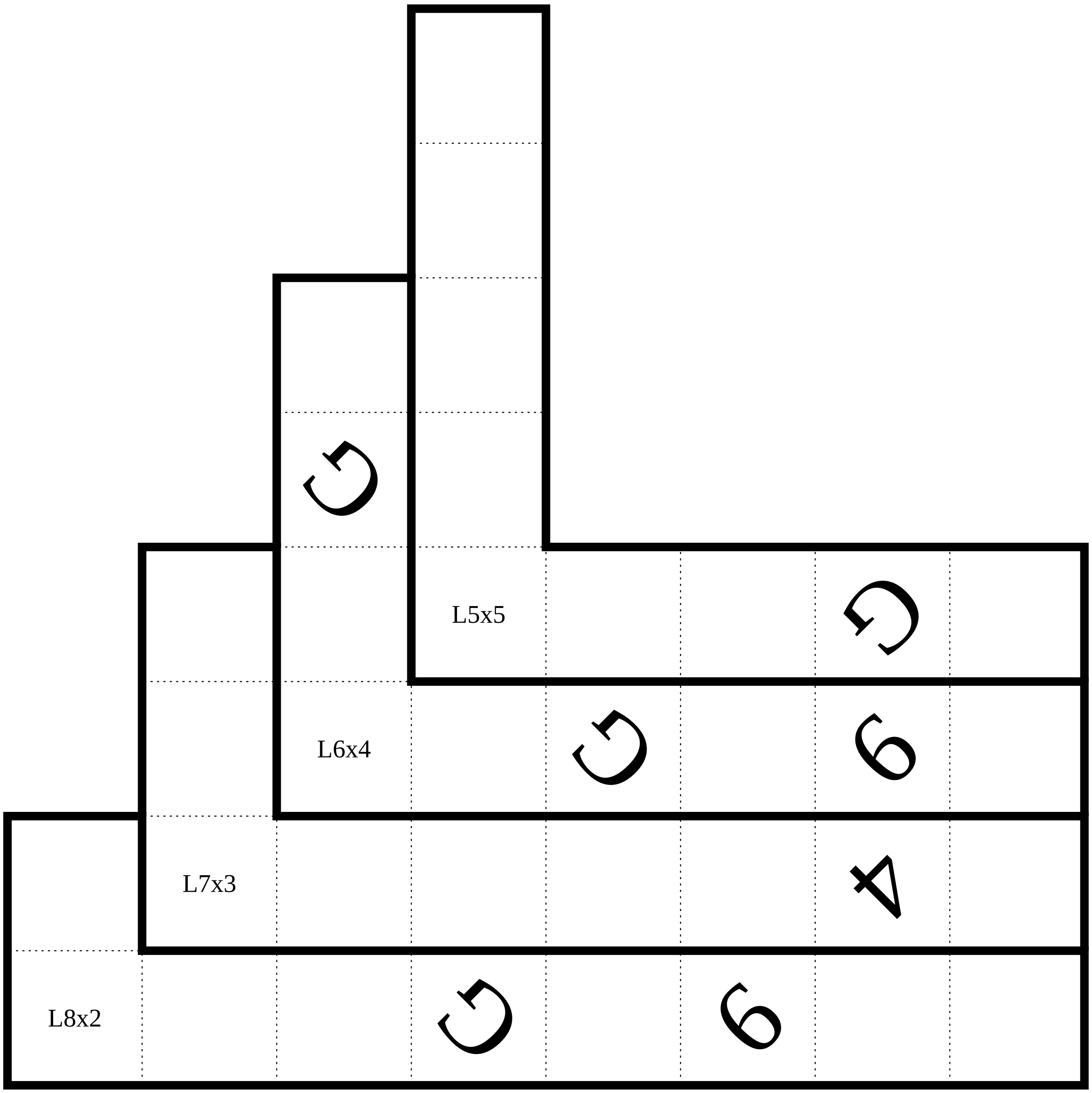

Figure 3: The four lominoes of area 9: $L8 \times 2$, $L7 \times 3$, $L6 \times 4$ and $L5 \times 5$.

To G4G10 ... and beyond

Suppose we take all lominoes with area 8, 9, and 10. We can calculate from Table 1 that their total area is 100. These pieces can be packed into a 10×10 square, and subsets can

Rectangle	No. of pieces	Solutions	Rectangle	No. of pieces	Solutions
9×9	10	9	9×6	6	2
9×8	9	16	8×6	6	10
12×6	9	15	10×5	6	4
8×8	8	3	11×5	6	13
11×6	8	7	8×5	5	4
13×5	8	1	10×4	5	11
9×7	7	6	8×4	4	2
8×7	7	6	8×3	3	2

Table 2: Squares and rectangles that can be packed using the ten piece Lomino81 set.

be packed into a 9×9 square, or 16 rectangles [2].

Is there a pattern here? What about lominoes of area 9, 10 and 11? Their total area is 131, too much for an 11×11 rectangle. But we can still build an 11×11 rectangle leaving one piece out (one of the lominoes of area 10). This puzzle is solvable, and has only 5 solutions.

In fact, an analogous puzzle can be extended indefinitely! In each case our goal is to make an $n \times n$ square from different lominoes of area $n-2$, $n-1$ and n. For $n > 10$, the total area of the pieces is greater than n^2, so we have to leave some pieces out. Figuring out which pieces to leave out is part of the puzzle.

These puzzles rapidly become too difficult to solve by hand, but make for good challenges for computer solvers, such as BurrTools [4]. Table 3 shows the number of solutions up to $n = 16$. Interestingly, the number of solutions is always relatively small, and there is no solution for $n = 16$. Although I have not been able to finish a complete run for $n = 17$, I have found two solutions by guessing the pieces that might work (Figure 4b).

Square size (n)	Lominoes of area	No. of pieces	Solutions Total	Solutions No nest	BurrTools runtime
9×9	7,8,9	10	9	2	2 seconds
10×10	8,9,10	11	5	2	7 seconds
11×11	9,10,11	12	5	0	3 minutes
12×12	10,11,12	13	28	2	10 minutes
13×13	11,12,13	14	7	0	1.2 hours
14×14	12,13,14	15	5	0	4.4 hours
15×15	13,14,15	16	2	0	17 hours
16×16	14,15,16	17	0	0	2.5 days
17×17	15,16,17	18	≥ 2	?	incomplete

Table 3: Number of solutions to lomino packing problems of increasing size. BurrTools runtimes are on a 2 GHz PC with 1 GB of RAM.

In Table 3, you will find a solution column heading of "No nest". We call two lominoes nested when one is placed inside another, forming a "L" shape that is two squares wide, as in Figure 4a. A "no nest" solution is one that contains no pair of nested lominoes. No nest solutions may be preferred for aesthetic reasons, but have been found only for $n = 9, 10$ and 12.

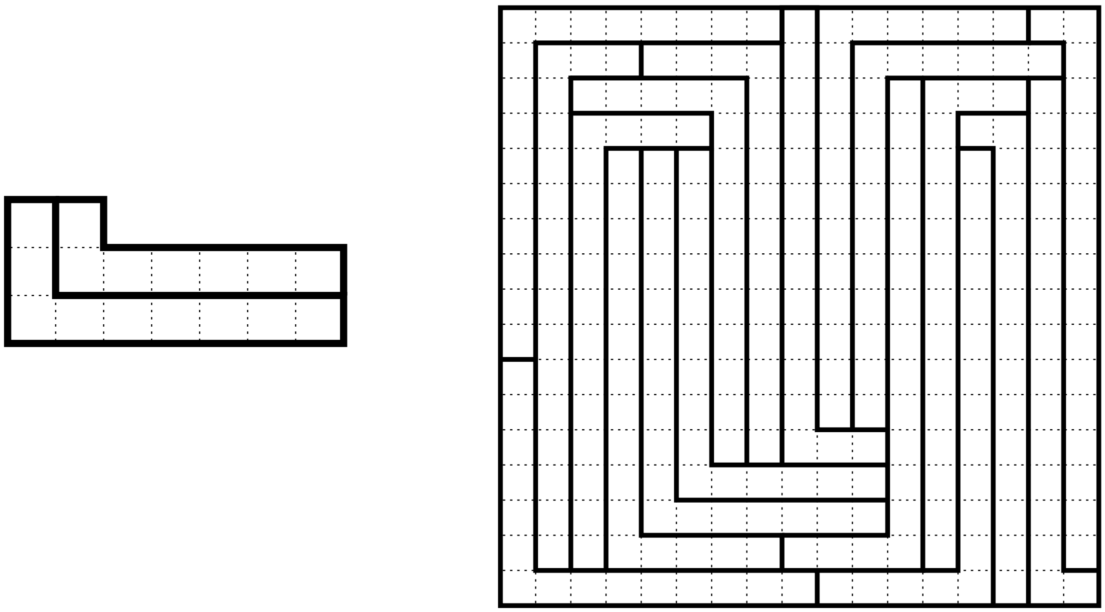

Figure 4: (a) A pair of nesting lominoes, (b) 18 different lominoes of area 15, 16 and 17 pack a 17×17 square (found by BurrTools [4]). Can you find the nesting pair?

References

[1] Alan H. Schoen, A Potpourri of Polygonal and Polyhedral Puzzles, in *Homage to a Pied Puzzler*, edited by Ed Pegg, Jr, Alan H. Schoen and Tom Rodgers, A K Peters, 2009.

[2] Ishino Keiichiro, Puzzle Will Be Played, Lomino100 puzzle `http://www.asahi-net.or.jp/~uy7t-isn/Puzzle/Lomino/100/`

[3] `http://www.comcast.net/~gibell/g4g9/`

[4] `http://burrtools.sourceforge.net/`