Some Notes on Ball-Pyramid and Related Puzzles

[comments in red added by George Bell in 2013]

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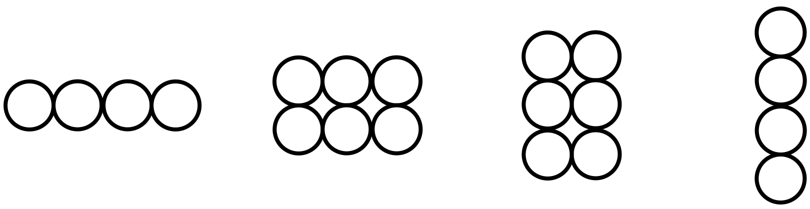
I have been fooling with ball puzzles for a dozen years now. Although I have made and sold puzzles commercially, this business venture has been minor. My primary interest has been in devising and solving puzzles. This is an attempt to describe some of them, and to record thoughts for future experiments. I also hope to interest mathematicians in the serious study of sphere packing-stacking puzzles.

Feel free to make any of the suggested puzzles for your own use. If you describe any in a publication, I ask only that you acknowledge the source. Please contact me before using any commercially.

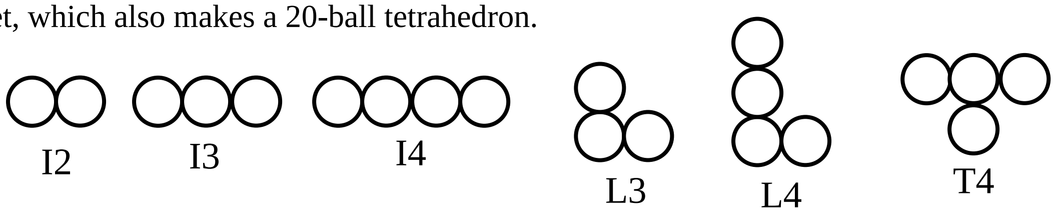
Warning: I give puzzle descriptions, not solutions. If you want to display these puzzles, you must solve them yourself. Some are difficult. [BurrTools can solve many of these puzzles.]

1. Ordinary Pyramid Puzzles

I began experimenting with pyramid puzzles circa 1972. At that time there was a puzzle being sold in America called "Tuts Tomb" [1]. It consisted of 4 pieces that form a tetrahedron, this was the first ball puzzle I had seen. It may have been the first commercial ball puzzle. The inventor was not identified on the package [2].



Although "Tuts Tomb" is too easy, I was intrigued by the orthogonal arrangements of balls within apparent isometric assemblies. I experimented with other pieces, and discovered this set, which also makes a 20-ball tetrahedron.



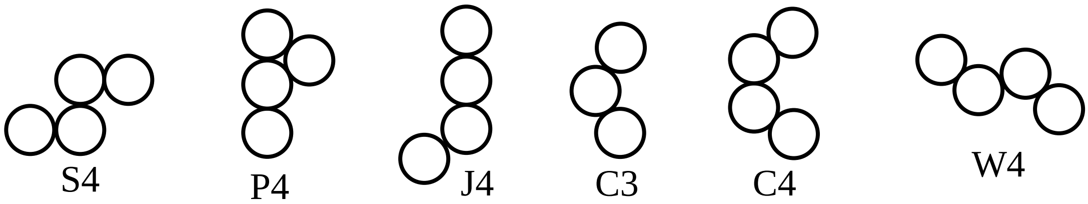
There is essentially only one solution. The tricky feature is that the orthogonal pieces are not in parallel planes, as in "Tuts Tomb". These 6 pieces also make a pyramid with a 3x4 rectangular base. I devised other 20-ball pyramid puzzles and concluded this one was best. I categorically eliminated those with isometric pieces only. [An isometric piece is based on a lattice of equilateral triangles and involves only angles of 60, 120 and 180 degrees.]

In 1974 I decided to produce the above as "Perplexing Pyramid", taking a clue from a burr puzzle being sold at the time. It was made of clear plastic, and the included bubbles (a natural consequence of the molding process) were attractive [3].

In 1979, I chose to sell two larger pyramid puzzles. "Big Pyramid" contains 30 balls on a 4x4 square base. "Giant Pyramid" is a 35-ball, size-5 tetrahedron. We distributed several versions of each before discovering the following with unique solutions.

Big: I4, T4, T4, S4, S4, S4, L3, L3 (8 pieces)
Giant: P4, J4, C4, W4, L4, L4, L4, L4, I3 (9 pieces)

This particular version of "Giant" was devised by my brother, Jerry. It is diabolical.



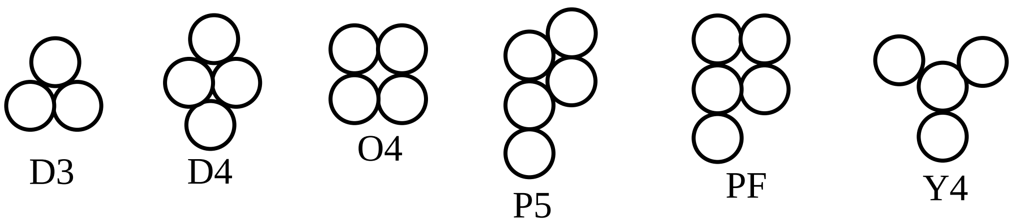
Although Jerry lives in New York and does not take part in manufacturing, we correspond about solving. Here are a few puzzles we solved over the years. I have tried to show some use of each of the different pieces, and unlikely placement of certain ones.

L4	I4	S4	T4	C4	P4	J4	W4	L3	I3	C3	I2	Pieces	Solutions†
1	1	1	1	1	1	1	1	1				9	64
1	1	1	1	1	1	1	1		1			9	167
	2	1	1	1	1	1	1	1				9	224
3	1			1	1	1	1	1				9	31
				2	2	2	2		1			9	6,695
	1		3	1	1	1	1		1			9	11
	1	3		1	1	1	1	1				9	5
						8				1		9	32
5	1				1		1		1			9	4
4‡				1	1	1	1		1			9	1
1				4				5				10	2
1				1	1	1	1	5				10	58
6									3		1	10	4

Table 1: Combos that make 35-ball size-5 tetrahedron, "Giant Pyramid"
† - Column added using BurrTools; ‡ Row added, puzzle mentioned in text.

L4	I4	S4	T4	C4	P4	J4	W4	D4	Y4	L3	I3	C3	D3	I2	Pieces	Solutions†
3		1	2								2				8	4
3	1		2							1	1				8	48
1	1	1		1	1	1						2			8	3,605
2	1	2	2											1	8	1
	1		1	1	1	1	1			1	1				8	621
2		1	3							1	1				8	14
	1			1	1	1	1	1				1	1		8	1,145
2	2		3											1	8	1
1				1	1	1		1	1	1	1				8	310
	1‡	3	2							2					8	1
	1	3								2	2			1	9	32
										10‡					10	2

Table 2a: Combos that make 30-ball size-4 square base, "Big Pyramid"
† - Column added using BurrTools; ‡ Row added, puzzle mentioned in text.



L4	I4	S4	T4	C4	P4	J4	W4	D4	O4	P5	PF	Pieces	Solutions†
1		1		1	1			1			2	7	22
				1	1	1	1	1		1	1	7	44
		3			1	1				1	1	7	1
1				2	2					2		7	24
				2	2				1	2		7	7
					5					2		7	28

Table 2b: More combos that make 30-ball size-4 square base, “Big Pyramid”
† - Column added using BurrTools

2. From Cubes to Box Packing

When I first became interested in puzzles, circa 1970, the SOMA cube was popular. Taking up from there, I experimented with puzzles I called "anti-cube". These consisted of pieces made of unit cubes joined edgewise. With 3 or 4 units per piece, a large family of puzzles is possible. Unable to devise a practical method of producing "anti" puzzles, I eventually quit studying them [4], [5].

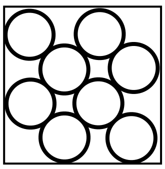
In 1973, I discovered the "hybrid" cube. Join unit cubes face-to-face and/or edge-to-edge. There are precisely nine different 3-cube pieces. Like SOMA these form a 3x3x3 cube and many other shapes. Playing with this puzzle led me to investigating box filling puzzles using ball pieces [6].

Cubes joined face to face can be replaced by touching balls. Cubes joined edgewise can be replaced by balls joined by a dowel with center-center spacing = $\sqrt{2}$. There is one degeneracy in the process so 2 ball-pieces are alike. The resultant assembly of the 9 ball-pieces to a cube like structure was not free standing, so I made a box.

Direct substitution of balls for cubes produces simple cubic packing, this is a less efficient packing than face centered cubic (FCC) the latter is identical to the spherical close packing in pyramids [7]. It is also possible to create box filling puzzles having a packing of intermediate efficiency known as body centered cubic (BCC) packing, I made puzzles with each type, all are interesting, but I will only describe those with FCC here. A different efficient packing known as hexagonal close packing does not lead to nice box puzzles, but it allows interesting pyramids which I will describe below.

3. Box Filling Puzzles - Face Centered Cubic Packing

Much of our (Jerry joined me here) early work with box filling puzzles was devoted to cubes containing 32 balls, and showing the following FCC arrangement on all faces.



Here are sets of 8 4-ball pieces which fill the box. Some of them are fairly easy puzzles. There are at least 12 different solutions to the last set.

L4	I4	S4	T4	C4	P4	J4	W4	D4	Y4	Pieces	Solutions†
		2	2		2			2		8	103
			4					4		8	1
8										8	3
						4		4		8	3
4								4		8	6
	2	2						4		8	1
1	1	1	1	1		1		1	1	8	8
2		4							2	8	3
1	1	1	1	1	1	1	1				102

Table 3: Combos that fill a 32-ball cubic box
† - Column added using BurrTools

This puzzle is a good size and further experimentation should be made. From the 11 simple planar 4-ball pieces available, myriad combinations of 8 pieces exist. The ideal, of course, is one with a unique solution. Surprisingly, 8 P4's do not fit, although 8 L4's do. It’s an ‘ell of a puzzle!

To generalizes for boxes with FCC packing, the number of balls to fill are
$n = (abc)/2$ or
$n = (abc + 1)/2$ when a, b, c are all odd
In the odd case, there is always an inferior packing that apparently fills the box with one less ball. The smallest puzzle that illustrates this is a=b=c=3 (n=13 or 14). a=5, b=c=3 (n=23 or 24) is more interesting.

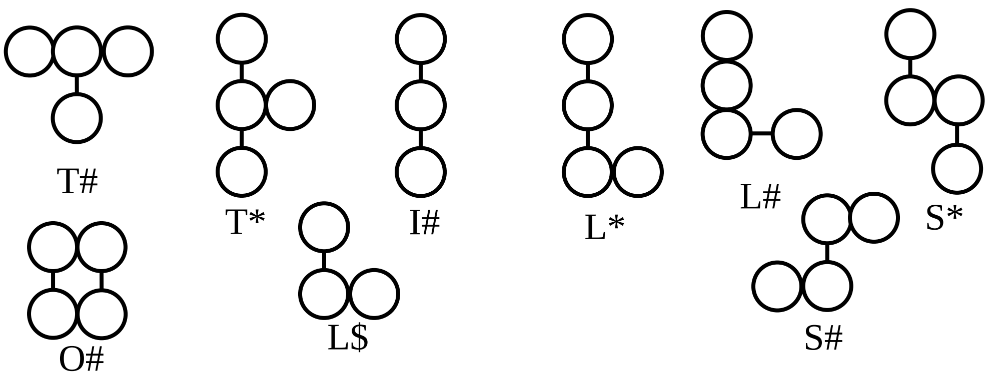
Note that for a=5, b=4, c=2, the box holds 20 balls. The parts of "Perplexing Pyramid" do not fit, but you can find several sets that do make a pyramid and fill the box (see Section 15).

There is no box of this type that holds all 11 4-ball planar pieces. a=11, b=4, c=2 holds 44 balls, but the pieces do not fit.

4. The Other Plane

Pictured below are pieces made with dowels. I call these "contraplanar" pieces. In spherical close packing there are six contraplanes, as well as three orthogonal and four isometric planes. It is very interesting to see how these planes coexist [8].

For practical reasons, my inventory of "contras" is limited to parts which will not be distorted by rotation about the dowel. Here are a few combinations using contras in the size-5 tetrahedron.



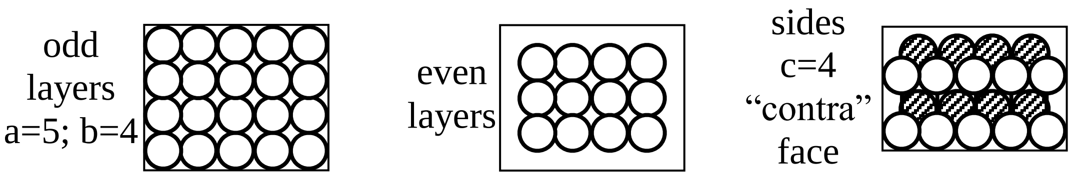
L4	S4	T4	C4	P4	J4	W4	L3	I3	T#	T*	I#	L*	L#	L$	O#	Solutions†
4									4		1					1
3							1		4	1						1
2	1						1		4			1				1
2			1	1				1	4							3
				1	1	1			2			1	2	1		25
1	1	1	1	1					3					1		5
1	1				1				3			2	1			0?
							5		5							2
				1	1				1			1		5	1	30

Table 4: Assemblies using “contras” in 35-ball tetrahedron
† - Column added using BurrTools

The last one has all 10 interstices filled. The next to last one has only two different pieces. S* is too awkward to use in this assembly. The interlocking potential of contra pieces is evident. In 1983 [9], I introduced a size-5 tetrahedron with this feature.

5. A Different Box

Using the contraplane as a face leads to an interesting set of box puzzles. The boxes show the following.



I refer to this aspect of spherical close packing as body centered prism (BCP) packing.

The number of balls in a box with dimensions a,b,c are

$n = abc - (c-1)(a+b-1)/2,$ c is odd
$n = abc - c(a+b+1)/2,$ c is even

There is a nice coincidence here. A box holding 30 balls with face centered cubic (FCC) packing (a=5, b=4, c=3) has almost the same dimensions as one with BCP packing (a=4, b=3, c=3). The boxes are almost alike, but the wrong orientation will not allow filling of either [10]. The size-4 shape based pyramid also has 30 balls. Here are four combinations of pieces that fill all three.

Set #1: P4,W4,I4,L4,S4,T4,I3,D3 D4,C4,J4,L4,S4,T4,L3,C3
Set #2: D4,C4,I4,L4,S4,T4,I3,C3 P4,J4,W4,L4,S4,T4,L3,D3

The 16 pieces in set #1 are the same as in #2. Take this as a 60-ball puzzle. Divide the 16 pieces so that each 30-ball subset makes all 3 assemblies. The above are then 2 different solutions.

Something else about this kind of box packing. For a=b=3, c=5, the box holds 35 balls. but now invert the layering along the "c" dimension, instead of 9,4,9,4,9 balls per layer, use 4,9,4,9,4 layering. This way 30 balls apparently fill. We have "left out" 5 balls. Here is a set with one 5-ball piece that can be left out: L4,L4,T4,C4,W4,P4,L3,C3,P5

One more variation. Stagger the layers. Make bottom layer 4x4 orthogonal array, next 3x4, and top 4x4. we now have a box holding 44 balls. Just what we need for the complete set of 4-ball ordinary planar pieces, and they do fit. Now, invert the layering and the box seems full with only 40 balls. Can we leave out any one of the 11 pieces?

6. Bigger and Harder

The number of balls in pyramids is

$n = s(s+1)(s+2)/6$ for tetrahedrons
$n = s(s+1)(2s+1)/6$ for square based pyramids

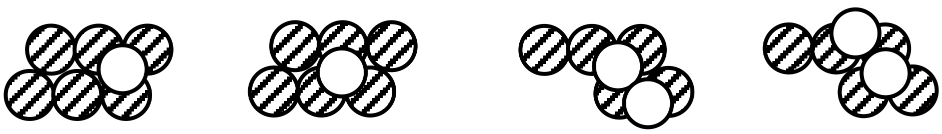
size	2	3	4	5	6	7	8	9	10
tetrahedron	4	10	20	35	56	84	120	165	220
square base	5	14	30	55	91	140	204	285	385

Those in any square base pyramid is the sum of those in two tetrahedrons. This can be derived from the formulae. There are precisely 33 planar 5-ball pieces and they assemble to a size-9 tetrahedron. This puzzle and a solution was described in a Japanese journal in 1972 [11]. I distributed additional solution(s) in 1984 [12].

There are 11 orthogonal planar pieces. I have not been able to assemble them to a size-5 square base pyramid. [I get 12 orthogonal 5-ball pieces (corresponding to the 12 pentominoes), but perhaps he is excluding I5? In any event, BurrTools indicates the assembly is impossible with any subset of 11 orthogonal pieces.]

165 is 3 times 55. Can the 33 pieces be divided so they make 3 size-5 square base pyramids? I think so. But this is probably very difficult. [BurrTools has verified that this assembly is possible.]

Note that 35, 56, 84, 91 and 140 are all multiples of 7. There are many ways to define a 7-ball set. Planar pieces seem impractical. Here is one puzzle using 3D pieces. These 4 different pieces plus their mirror images assemble to a size-6 tetrahedron (see Section 10).



All eight parts of this puzzle are made by joining P5 and I2. I have made some larger assemblies from this set, including combinations that make square base pyramids and divide into two smaller tetrahedrons (91=56+35).

There are 14 4-ball non-planar pieces [13] these might make a 56-ball size 6 tetrahedron. I have not done that yet. [BurrTools finds 1,388 solutions]

7. Low Profile Pyramids

Using the contraplane as a base, we get a group of pyramid puzzles numerically equivalent to square or rectangular based ones. The ordering is the same, but one horizontal dimension is stretched out and the height is lowered. I have experimented with 30-ball and 55-ball equivalent square base pyramids. They are fun. A 3-ball assembly can be incorporated in the box exchange concept described above. I leave finding combinations to the reader (see Section 1).

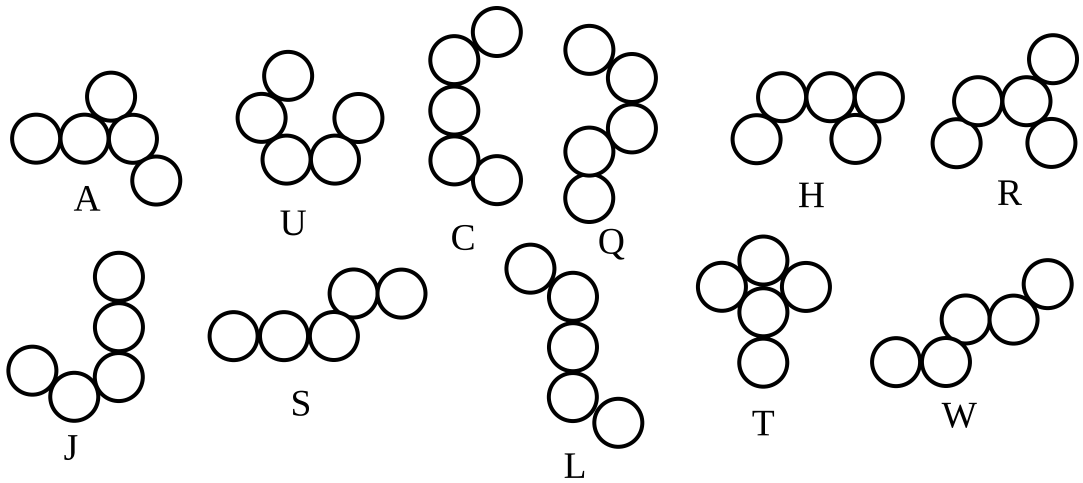
8. Size-5 Tetrahedrons and the Computer

I use a home computer and compiled basic to solve assembly puzzles. Having an algorithm, one can apply it to different puzzles, but poly-ball puzzles take more computer time than poly-cube puzzles because of the every-which way pieces fit in. Here are some results for the 35 ball tetrahedron.

Excluding the straight piece, there are 21 isometric planar 5-ball pieces. Taking them 7 at a time, there are 116,280 potential size-5 tetrahedron puzzles [14]. After computer solving several and recognizing pieces that tend to cause multiple solutions ("easy" pieces), I entered descriptions of 11 "hard" pieces and asked for all solvable combinations. Of the 33 possible, only the following "make it". Most have unique solutions.

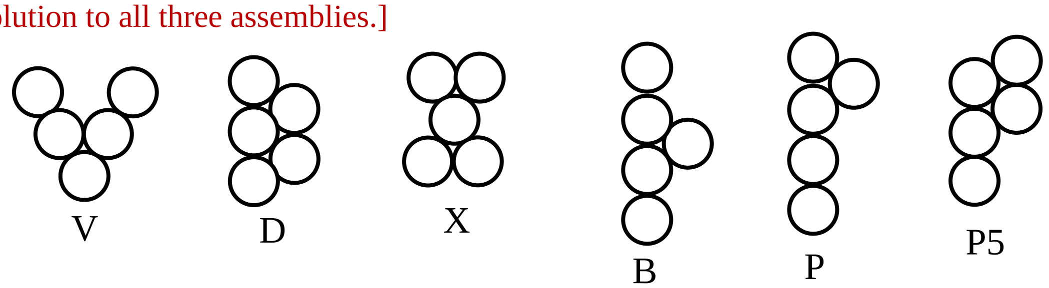
A	U	C	Q	H	R	J	S	L	T	W	Solutions†	55-Pyramid Solutions‡
1	1	1	1	1		1				1	1	0
1		1	1	1	1		1		1		3	4
1	1		1	1		1	1	1			2	1
1	1	1	1	1			1	1			0?	3
	1	1	1	1			1	1	1		2	0
	1	1		1	1	1	1	1			1	0
1		1		1	1	1	1		1		1	2
1	1	1		1			1		1	1	2	0
		1	1	1	1	1		1	1		1	0
		1		1	1	1	1	1	1		2	0
	1	1		1		1	1	1	1		1	0
		1	1	1	1			1	1	1	1	0
		1	1	1	1		1	1	1		1	0
1		1	1	1	1	1			1		1	0
1		1	1	1		1			1	1	1	0
		1	1	1	1	1	1	1			1	0
			1	1	1	1	1	1		1	1	0
	1		1	1		1	1	1		1	1	0
		1	1	1			1	1	1	1	1	0
1		1	1	1	1		1			1	2	2
1	1	1	1	1	1		1				2	7
1		1	1	1	1	1	1				2	10
1		1			1	1	1		1	1	1	0
1			1		1	1	1	1	1		1	0
1	1	1	1		1		1		1		0?	0
1		1	1		1	1	1		1		2	2
	1	1	1		1		1	1	1		0?	0
			1		1	1	1	1	1	1	1	0

Table 5: Sets of 5-ball planar pieces. Seven make a 35-ball tetrahedron.
† - Column added using BurrTools; ‡ Solutions for 55-ball pyramid adding V,D,X,B.



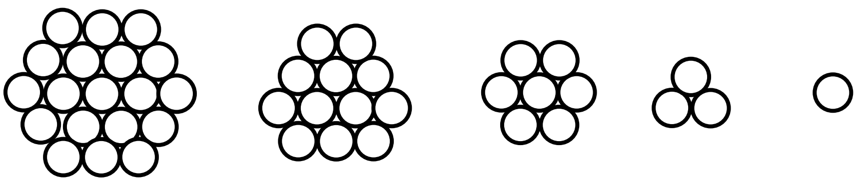
Solving these 28 puzzles should be challenge enough for anyone. But, if you want a truly fiendish puzzle, note that there is only one combination that omits both Q and H. Therefore, given the remaining 9 5-ball pieces, A, U, C, R, J, S, L, T, W, there is only one way to assemble a size-5 tetrahedron.

Having the above, we can turn to another problem. Can we find a set of 11 planar isometric 5-ball pieces that make a 55-ball size-5 square base pyramid and divide to make 35-ball plus 20-ball tetrahedrons? I have not done this. But why not? Here is a set of four "easy" pieces that make the Size-4: V, D, X, B. Add them to any of the solvable size-5 combinations and the resultant should make the bigger guy, I think. [This works, for some cases! I added an extra column to Table 5 showing the results. I didn't find a set of pieces which had a unique solution to all three assemblies.]

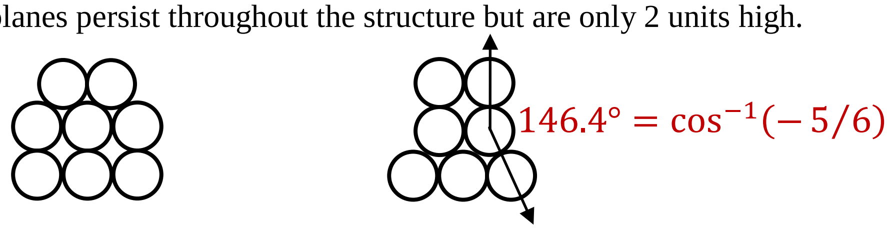


9. Hexagonal Packing in Pyramids

Lay out some balls (marbles?) in isometric (triangular) array. Add a second layer. A third layer can be added in either of two ways. If the balls in the third layer are directly over those of the first, we have hexagonal close packing. Keeping to this rule, build a pyramid using successively smaller hexagons in each layer. You will find that only every other layer can be a perfect hexagon, like this.



If you were careful to preserve the proper ordering, you will see these patterns on the sides. The orthogonal planes persist throughout the structure but are only 2 units high.

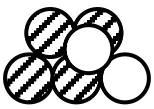


The number of balls in each layer and in each pyramid, counting the top are. [A general formula for a hexagonal pyramid with n layers is $\lfloor (n^2+1)(2n+3)/8 \rfloor$]

layer	1	2	3	4	5	6	7	8	9
balls / layer	1	3	7	12	19	27	37	48	61
balls / pyramid	1	4	11	23	42	69	106	154	215

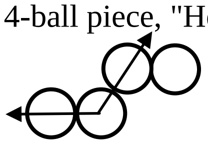
We can use regular ball-pieces to create a puzzle, but how do we force hexagonal packing? I have found two ways. One is to have a base with vertical poles at appropriate points. The pyramid must be built around and between these poles. The other way is to introduce pieces with internal angles found only in hexagonal packing. For example, make a 3-ball piece with the angle marked in the above drawing. The following also work.

Make a 6-ball piece by joining I2 to D4. I2 does not lie in a plane parallel to D4. [?] There are two mirror image resultants.



The "teepee" puzzle I sold to a few collectors in 1984 is a 106-ball hexagonal pyramid using the pieces just described. It appears formidable, but is not difficult if you study hexagonal close packing.

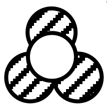
Here is another approach. Make this 4-ball piece, "Hex-S".



The angle shown is 109.5 degrees. [$\cos^{-1}(-1/3)$] Hex-S can only lie in a vertical plane. There are 3 such planes. Three is the most Hex-S pieces that can be incorporated into a 42 ball structure. Here are some puzzles.

3 of Hex-S, 4 of C4, 2 of W4, 1 each of D4, I2
2 of Hex-S, 5 of P4, 2 of D4, 2 of L3

And now to complicate the game even more, I introduce this 5-ball piece



The diagram shows 4 balls, a fifth lies directly below the top one. The piece resembles a "Top". Here is a puzzle, I would be a poor designer indeed if the "Top" fit on top.

2 of Hex-S, 1 Top, 1 each of C4, W4, S4, L4, D4, L3, I2, J4

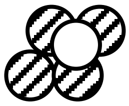
Note that there are 3 different S-type pieces. If you make this puzzle, please be sure the parts are accurate and strong. There will be a strong tendency for the solver to want to straighten some of them out!

10. Single Piece Puzzles

Returning now to spherical close packing. Four of each of the following isometric planar 5-ball pieces, A, D, H, P5, R and V, make a size-4 tetrahedron. The one made with "H" is the most interesting. It has two different solutions, one of which is interlocking.

Seven of each of D, or H, make the size-5 tetrahedron. In both cases, the size-5 triangle can be separated from the smaller tetrahedron.

Four of this 5-ball, 3D piece make a size-4 tetrahedron. This set is an analog to the well known dissected wood block tetrahedron puzzle.

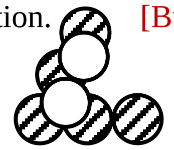


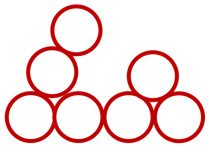
Five P4's make a size-4 tetrahedron. Ten L3' s make a square base pyramid. [This is the first mention of this excellent puzzle, later rediscovered by Torsten Sillke.]

The total number of balls in several of the larger pyramids is a multiple of five. Certainly some of them can be made from all of one 5-ball piece, other than the obvious D5. I have tried making size-8, size-9, and size-10 tetrahedrons from all H's. So far, I have succeeded only for size-10. [BurrTools finds all three problems are solvable.]

Here are a couple of almost single piece puzzles. The 56-ball size-6 tetrahedron can be made from 8 H5's plus 4 S4's or 4 W4's, or 2 C4's and 2 W4's.   In all cases, there is a symmetric [15] solution. In at least one case, an unsymmetric solution is also possible.

Although it seems that allowing 3D pieces, we should be able to find one 7-ball piece, 8 of which make the size-6 tetrahedron. I have not been able to do so. However, here is one that comes close. Four of these plus their mirror images do assemble. My computer confirms a unique solution.   [BurrTools finds exactly one planar piece which solves this problem.]





11. Four Piece Size-4 Tetrahedrons

There is a nice set of puzzles here. I generate them as follows. Arrange four J4's so that they are symmetrically disposed within the envelope of a size-4 tetrahedron. The longer arms should end in the vertices. The result will be a sort of cage with 4 separate open cells. A single ball in one of these cells touches 3 different J4's. Attach single balls to one or the other J4's uniformly, and 3 different puzzles can result. One puzzle has 4 planar all-alike pieces. Two have 2 sets of 3D mirror image pairs. Non-uniform attachment can produce 4 different pieces, some combos are interlocking. [In Wiezorke's Compendium, one of these puzzles is (incorrectly) identified as "Stan's Tetrahedron". It should be "Leonard's Tetrahedron".]

Similar puzzles can be derived from 4 each of W4, D4, P4, or C4. There will be some redundancy. In some cases the only possible results are planar pieces.

Four L4 pieces may be assembled within the envelope of a size-4 tetrahedron. They snap together. The 4 vertex cells will be empty. [This elegant puzzle was rediscovered in 1996 by

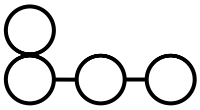
Bernhard Wiezorke, he called it “Blossom”.] A ball placed there can be attached to either of two pieces. This assembly might be used as a core for constructing polyhedrons or stars. Use your imagination.

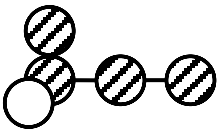
12. Strange Beasts

If we join a square base pyramid of size *s-1* to one of size *s*, we get an octahedron. The number of balls in each is

$n = s(2s^2 + 1)/3$

An octahedron has 6 vertices, remove one ball from each vertex of the 44-ball assembly [16] and we have a 38-ball truncated octahedron. The result looks like a "popcorn ball". The following 10 pieces make that structure and are interlocking: L4, Y4, L3, L3, L3

4 of these 

plus this one 5-ball non-planar beast. 

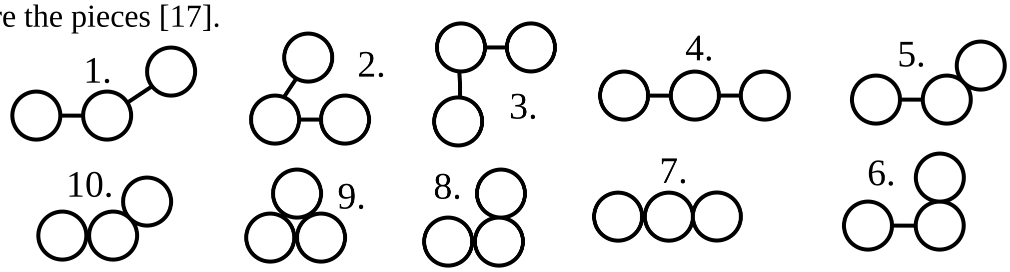
In this last piece, the dowel is attached to the central ball of an L3. This is one of a whole family of 3D pieces. They are nice looking "animals". I have not experimented with them very much.

Essentially, the above is a "burr" puzzle in balls. Here is another one. Add a 4-ball tetrahedron to each face of a 35-ball size-5 tetrahedron. The result is a stellated octahedron, this is the same as we would get if we added 8 4-ball, tetrahedrons to a 19-ball size-3 octahedron. Using mostly 5-ball contraplanar pieces, we can make a free standing interlocking structure that is even more burr puzzle like than the first one.

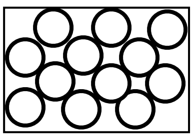
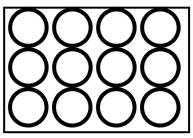
There are probably many other three dimensional geometric figures that be made from balls.

13. Box Packing 3-Ball Pieces

So far, I have recognized 10 different 3-ball puzzle pieces. Section 2 describes the equivalence between joined cubes and joined balls. The 8 different ball-pieces derived from the "hybrid" cube may be used in box packing puzzles based on a simple cubic array. Six of them may also be used in spherical close packing. Additionally, there are two 3-ball pieces used in spherical packing that do not fit into a cubic array. Note that the "contraplane" of spherical packing (Section 4) is the same as a diagonal plane in simple cubic packing. These are the pieces [17].



Since 8 of the 10 may be used in either packing, let’s make two boxes, each to hold 24 balls. Both boxes contain 2 layers of balls, the one with close packing has the FCC aspect.

FCC  Simple Cubic 

Although FCC packing is nominally more efficient than simple cubic, these boxes have almost exactly the same volume due to edge effects. Given 10 pieces, place as many as possible in either box. If everything is made accurately, this should be 8 for either.

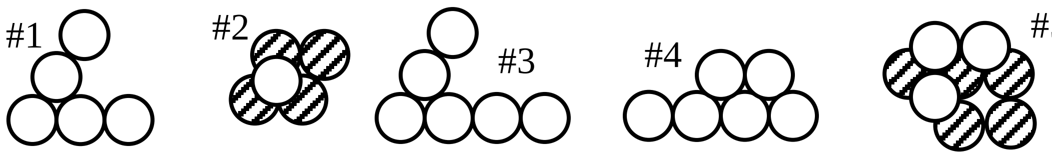
Now, let’s introduce a third box. This one of the type described in Section 5. Three layers: 2x5, 1x4 and 2x5 hold 24 balls. This is spherical close packing again. The volume is almost exactly the same as the first two, but one of the 8 3-ball pieces will not fit into this coffin shaped box. Here are two tricky combos that fit.

3, 3, 5, 5, 8, 8, 8, 8
3, 3, 5, 5, 7, 7, 8, 8

All in all, these 24-ball boxes make neat puzzles. They are easier than most of those described earlier, but instructive.

14. Minimum Piece Dissection

In Section 5 I point out that the number of balls in any square base pyramid is equal to those in two smaller tetrahedrons. The following five pieces make a 30 ball square base pyramid or a 20 ball and a 10 ball tetrahedron, I think this is a minimum piece dissection.



Actually, two each of pieces #1 and #2 plus a 10 ball triangle do the same thing, but the above set is more jazzy.

15. Teaching Sets

The following pieces add to 20 balls. The set includes right-angle and isometric, but no contraplanar pieces [18].

L4, I4, P4, D3, L3, I2

1.	Form a tetrahedron	Section 1
2.	Fill a box showing FCC packing	Section 2
3.	Make a pyramid with 3x4 rectangular base	Section 1
4.	Make a low profile pyramid	Section 7

The following pieces add to 30 balls.

L4, T4, J4, P4, P4, S4, D3, L3

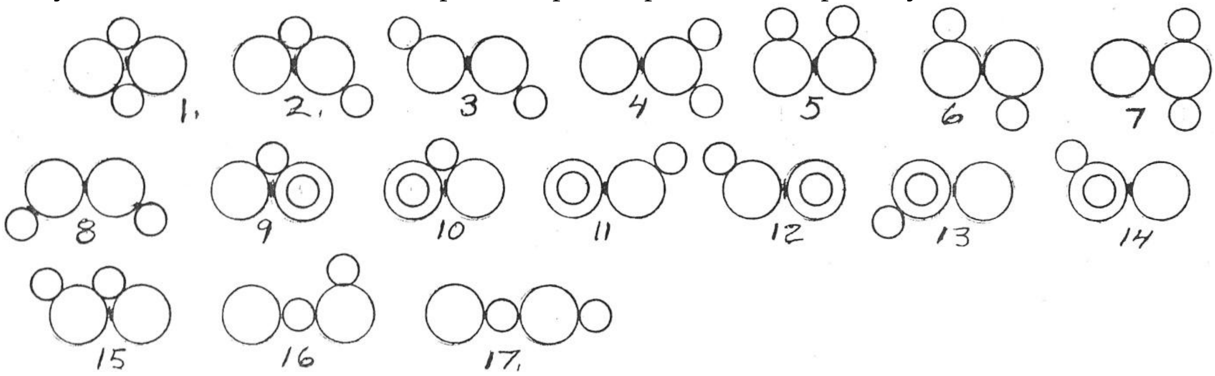
1.	Make a square base pyramid	Section 1
2.	Fill box with FCC packing	Section 5
3.	Fill box with BCP packing	Section 5
4.	Make a low profile pyramid	Section 7

[Note: This last set is similar to Gordon’s puzzle: WARP-30. WARP-30 uses the piece set: L4, T4, J4, P4, S4, C4, W4, I2, and has the same 4 tasks.]

Part 2 [These pages were evidently added later]

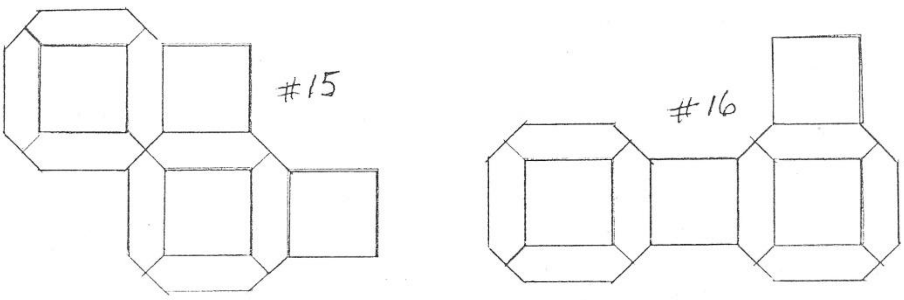
16. Box Packing Puzzles and Checkered Blocks

Section 2 discusses filling boxes using FCC packing. From that aspect of spherical close packing [19], the voids (interstitial sites) are apparent. Section 4 describes pieces made with dowels which go thru the I-sites in an assembly. The idea was to make interlocking puzzles. Let's do something different. Add smaller balls to the larger ones merely to fill, and in particular, to fill all I-sites in a box. Except in the odd case, a box with FCC packing has as many I-sites as balls. Here are 17 possible puzzle pieces with 2 primary balls and 2 I-balls.



Take the first 15 as a set. We then have 30 balls of each size. Alternately, add #16 or #17, and we have 32 balls of each size. [If the large balls have radius 1, the small balls have radius $\sqrt{2} - 1 \approx .414$]

Filling a box with balls of 2 different sizes is equivalent to building a checkered block of cubes. Instead of two colors, we have two sizes. Large cubes have all 12 edges beveled and join each other on the bevel faces. Small cubes are attached to normal faces of the large ones, and do not touch each other.



This is a good way to build a checkered cube puzzle because the solver cannot "cheat" on the hidden internal cubes. The assembly of 16 ball-pieces just mentioned is equivalent to a 4x4x4 block. Jerry Gordon has solved this puzzle (using #16). He has also solved the 15-piece, 3x4x5 block, in this form. I have no idea how many solutions there are. This assembly is too large for my home computer. 15 4-unit pieces make for a more complex problem than 12 5-unit pentacubes.

17. From Balls Back to Cubes or ...

A beveled cube is a truncated rhombic dodecahedron (TRDDH). The bevel face is an elongated hexagon. Join units on these bevel faces to make puzzle pieces. They assemble to pyramids or fill boxes in almost the same way as ball pieces. Interstitial voids are cubic.

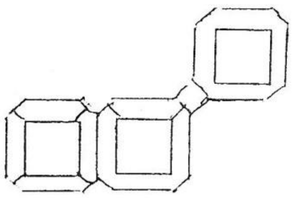
Beveling all 12 edges of a bunch of cubes is not the easiest way of making a puzzle, but try one. The 20-unit set given in section 15 makes a good sample. Assemble the RDDH pieces to the pyramids or pack a box just as described for the ball puzzle. The comparison with ball puzzles is very interesting. You can also use complete RDDH if you like, but beveled cubes are less demanding of your woodworking skill [21].

As I just said, pieces made with RDDH's assemble like ball pieces, except there is a difference. Join 3 RDDH (truncated or not) to form the equivalent of a 3 ball piece, and you will find that only one side accepts a 4th unit centrally. Spherical close packing is forced by the RDDH. From the RDDH D3, we see that two different P4's are possible. I also find that there are two different J4's and surprisingly two different X4's.

There are 11 different planar 4-ball pieces, hence, 14 different planar RDDH pieces. 14x4=56, and there are 56 units in a size-6 tetrahedron. In spite of the labor involved, I could not resist making these 14 pieces and they do assemble to a tetrahedron [22]. A box packing puzzle of the type described in Section 3 is also possible.

An interesting note here. Section 2 describes converting pieces made of cubes joined edgewise to pieces made of balls. In one case, the distinction between two different pieces was lost in the conversion. Now, in converting from balls to RDDH, I reverse the process. Certain pieces which are reflexible when made of balls evolve into a pair of enantiomorphs when made from RDDH. One caution ... converting the 20-ball puzzle just described from balls to RDDH works OK because it contains only one piece that exists in enantiomorphic form. Not all puzzles that are solvable as balls are solvable as RDDH. To be more precise, any single assembly can be converted to RDDH, but other assemblies with the same ball-pieces may not be possible with RDDH.

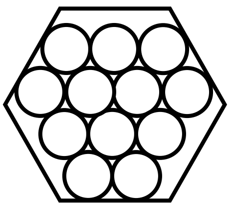
Incidentally, the beveled cube would be ideal for making "anti-cube" and the "hybrid" puzzles. Just add spacers.



Unfortunately, I don't know how to make beveled cubes economically in the moderate quantities warranted for these various puzzles.

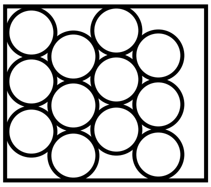
18. More on Hexagonal Close Packing. A More Complete Set.

Although I did not think so when starting this study, good HCP box puzzles are possible. The problem is that they need to contain a fairly large number of balls in order to fully develop the hexagonal ordering. The 4-ball planar piece described as "Hex-S" in Section 9, adds to the 11 we know in spherical packing to make a 12 piece set. Here is an assembly of 48 balls that makes for a "tough" but elegant puzzle. Lay out 12 balls like this.



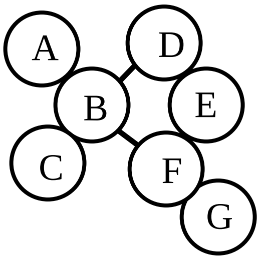
Rotate the pattern 60 degrees to make the 2nd layer. The 3rd and 4th layers are directly above the 1st and 2nd respectively. The envelope is a true hexagon. Hex-S must stand in this box. L4, T4, S4, and O4 can only lean. I4 and X4 must lie flat. P4, C4, D4, W4 and J4 complete the set. I was unable to solve this puzzle off hand, so went for the computer. We found 7 solutions. 12 pieces in 48 cells do not usually form a difficult puzzle, but the change in perspective is tricky. Describing the ways each piece can be oriented in the odd or even layers, made for a very interesting computer program.

If you make the following pattern for 4 layers, and stack them properly, you get a rectangular box puzzle using the full HCP set. It has quite a few solutions it is not as elegant as the hexagonal box puzzle, but still plenty "tough".



19. Contraplane in Hexagonal Packing

In Section 4, I describe a "contraplane" for spherical close packing, here is an equivalent for hexagonal packing. The distances between ball centers (legs of the triangle) are 1, $\sqrt{2}$, and $\sqrt{3}$. Hex-S lies in this plane. There are 3 sets of these planes in an assembly.



$\angle$ABD = 90^o
$\angle$DBF = 70.5^o
$\angle$DEF =109.5^o
$\angle$ABF = 160.5^o

Refer to Section 13 on 3-ball pieces. Pieces #5, 7, 8, 9 and 10 fit into HCP in the same way they fit into FCC packing. #6 is present in the above contraplane as ABD. Three 3-ball pieces are unique to that plane, ABC, ABF, and DBF. There is also a 3-ball piece that I have neglected so far. The angle shown in the second drawing of Section 9 is 146.4 degrees. A plane thru the three ball-centers of this piece passes thru centers of identical pieces. This is an interupted plane. This 3-ball piece is the only one possible in that plane. We now have 10 different 3-ball pieces that fit into HCP. It shouldn't be too difficult to create a box packing puzzle.

20. A Rhombic Dodecahedron in Hexagonal Packing?

The rhombic dodecahedron is not a uniform polyhedron [23]. A RDDH has 12 identical rhombic faces, but 4 rhombi meet at each of 6 vertices and 3 meet at each of 8 vertices. Orient a RDDH so that a vertex where 3 faces meet is on top. There will then be 6 vertical faces. Dissect horizontally, rotate the upper part 60 degrees with respect to the lower, and rejoin. The 6 vertical faces are now trapezoids. This polyhedron is a "trapezo-rhombic dodecahedron" [24]. It is space filling in HCP. Producing puzzles from these polyhedrons might prove interesting. I do not know. Off-hand, I would avoid making the actual polyhedrons, maybe by using hexagonal prisms with appropriate cuts and knobs.

21. Simple Cubic Packing and the Rhombic Dodecahedron

Although the internal ordering of the atoms is simple cubic (like common salt), common garnet is often found as RDDH crystals. It is interesting to see how this comes about. Maybe there is some puzzle potential here. The following sketch shows an assembly of atoms in cubic array that has a RDDH envelope. I am not a good enough artist to draw this for spheres, so illustrate with cubes. A plane segment joining the centers of cubes A, B, C, D, E is a rhombus. It may be a little difficult to accept this angular construction as a garnet crystal, but when it grows to have 10^{23} atoms, the faces will smooth out.

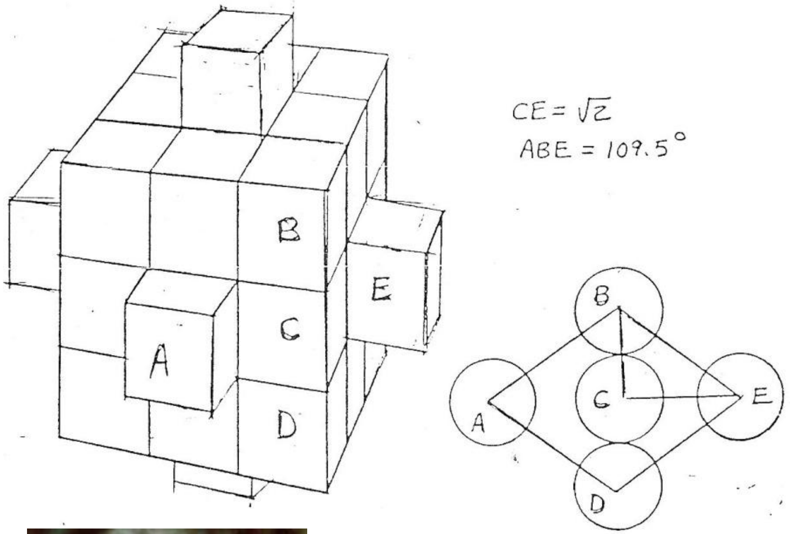



Photo courtesy Jerry Slocum

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CHICO, CA
95928

Footnotes

1. Made by Mag-Nif Co.

2. In 1985, Jerry Slocum sent me a copy of instructions for a 20-ball pyramid puzzle invented by Piet Hein of Denmark. It had a copyright date of 1970. Later, Slocum searched Martin Gardner's files and found the same instructions dated 1967. Gardner made brief mention of this puzzle in a Scientific American column. I don’t remember when, nor do I remember when I read the column. The puzzle has the 6 isometric pieces: D3, D3, I3, C3, D4, P4. [The February, 1973 Scientific American contains this reference, which is reproduced in Martin Gardner's book: "Mathematical Carnival" in the postscript section for Chapter 18. The name of the puzzle was "Anagog", marketed by Hubley Toys. Gardner also mentions that the six pieces can be assembled into either a 20-ball tetrahedron or two 10-ball tetrahedrons.]

3. "Prism Puzzle" by Pacific Games Co.

4. The book "Polyominoes" by Solomon Golomb, published in 1965, makes brief mention of joining squares cornerwise, but they were not included in his original definition. Martin Gardner described Golomb's polyominoes in 1957 and Hein's Soma Cube shortly after. Soma was invented in 1936.

5. My Thinkamajig puzzle is a 2D version of squares joined cornerwise. It was sold by Skor-Mor in the early 1970's. Later, I produced and sold two SOMA-type puzzles made of cylinders joined endwise and tangentially.

6. Matti Linkola of Finland informed me that the 9-piece cube puzzle I discovered as the hybrids was described by Jorgen Lou of Denmark in a 1972 patent application (we dont know if it was granted). Although I was working with cubes joined edgewise in 1970, I did not recognize this combo until 1973, so his discovery was previous to mine.

7. Only metals crystalize in either of the two close packings. Crystaline refers to the internal arrangement of atoms as different from the macroscopic crystal shape. Garnets are mixed irons calcium, aluminum or magnesium silicates. They have simple cubic packing, but form many different crystals. Almandanite is found as rhombic dodecahedrons.

   Here are the way some metals crystalize, those with HCP are less ductile than the others.
   FCC: aluminum, copper, gold, lead, silver
   HCP: magnesium, titanium, zinc
   BCC: iron, chromium
   Crystals are known with mixed hexagonal and spherical packing. Any puzzles here?

8. Which is what we learned in the discussion of X-ray crystallography in chemistry class (if we were paying attention). Place a ball-tetrahedron face to face with another and we have an example of "twinning". Place the size-4 tetrahedron face-to-face with the size-4 square based pyramid and we illustrate the relationship that was discussed by the American press in regard to the wrong answer in the SAT test a few years ago. [for a good discussion of this, see Mathematical Puzzles, a Connoisseur's Collection, by Peter Winkler, A K Peters, 2004, p. 43-44]

9. Puzzle party at Jerry Slocum's home in Beverly Hills [1983, IPP#6. This was the first IPP attended by Leonard Gordon].

10. Be sure the box is made accurately and strong. Trying to push a ball into a space just a little too small exerts a strong force on the wall. If you make a clear box of acrylic plastic, use small screws as well as glue. My experience is that a transparent box does not help in solving. By the time you become familiar enough with the pieces to be able to solve the puzzle you will be remembering where they are. Cardboard boxes held together with glass reinforced tape work well for me.

11. This information was provided by Kiyoshi Takizawa in 1983. A size-9 puzzle built by William Strijbos of Holland contained mostly all different 5-ball planar pieces, plus some 4-ball pieces.

12. Slocum's puzzle party. [Jerry Slocum's detailed IPP notes indicate this must have been IPP #8 in 1986, and that both Gordon brothers attended IPP #8.]

13. Enumerated by Takizawa.

14. One such, devised by John Bird, was sold in the 1970's by Skor-Mor as "Cannonball". It is difficult enough for the casual solver. My computer found 29 solutions. [In the notation of Section 12, the 7 pieces are: B, D, J, R, V, P, P5. BurrTools finds 29

solutions. Mike Reilly teamed with John Bird in 1972 to present this puzzle to Skor-Mor. Mike told me in an email it is his design.]

15. Not quite symmetric. There are 4 apex sub-assemblies. Each is made of 2 H's. Two subassemblies are enantiomorhic to the others. I think this is a fundamental requirement, but cannot prove it. In those single piece puzzles which I have found, the piece is reflexible.

16. The 44-ball octahedron can be made of all 11 4-ball planar pieces. This was sold as "Tetra" puzzle in Japan.

17. Piece #5 of this group is one of what I call "extended" orthogonal pieces. #3 and #4 are part of this set as well as part of the contraplanar set. I count 17 4-ball "extended" pieces. Will 16 of them fit into a cubic box?

18. As I mentioned in Section 3, the parts of "perplexing pyramid" do not fit in this box. They also do not make a low profile pyramid. On the other hand the parts of this set are much easier to assemble to the tetrahedron and rectangular base pyramid. This is probably general. We can make a puzzle harder by choosing parts specifically for it.

19. Some writers always refer to spherical close packing as FCC. The two packings are then FCC or HCP.

20. The smaller atoms, hydrogen boron, carbon and nitrogen enter the interstices of other elements. For example low carbon steel is an interstitial solution of carbon in BCC iron. The number of interstitial atoms must be small, otherwise compounds and new structures form. Dispersions of these compounds is what makes high carbon steels hard.

21. Stewart Coffin has made several puzzles from RDDH and beveled cubes (TRDDH). He has also used other building blocks. We can make pyramid or box packing puzzles from all kinds of units, space filling or not. However, many resultants are merely 3D jigsaw puzzles.

22. I made the bevel face comparatively small so that there are large cubic voids. They are all hidden in the assembled pyramid, just like crypts in the real pyramids.

23. The true dodecahedron has pentagon faces, and is uniform. It is not space filling.

24. The subject of ball packing and RDDH is discussed in "Introduction to Geometry" by Coxeter, and in "Mathematical Recreations and Essays", by Rouse Ball. However, I find it impossible to visualize any of these polyhedrons from the drawings. One must make a model.

I will enjoy hearing from anyone who is interested in ball puzzles or ball geometry, I will also enjoy seeing descriptions of other puzzles. But please, no new variations of those I have described. Cataloging those could become overwhelming.

I will be very interested to learn of any work with ball puzzles prior to 1970.

Leonard Gordon