Nova Plexus and Tetra Cage

by George Bell

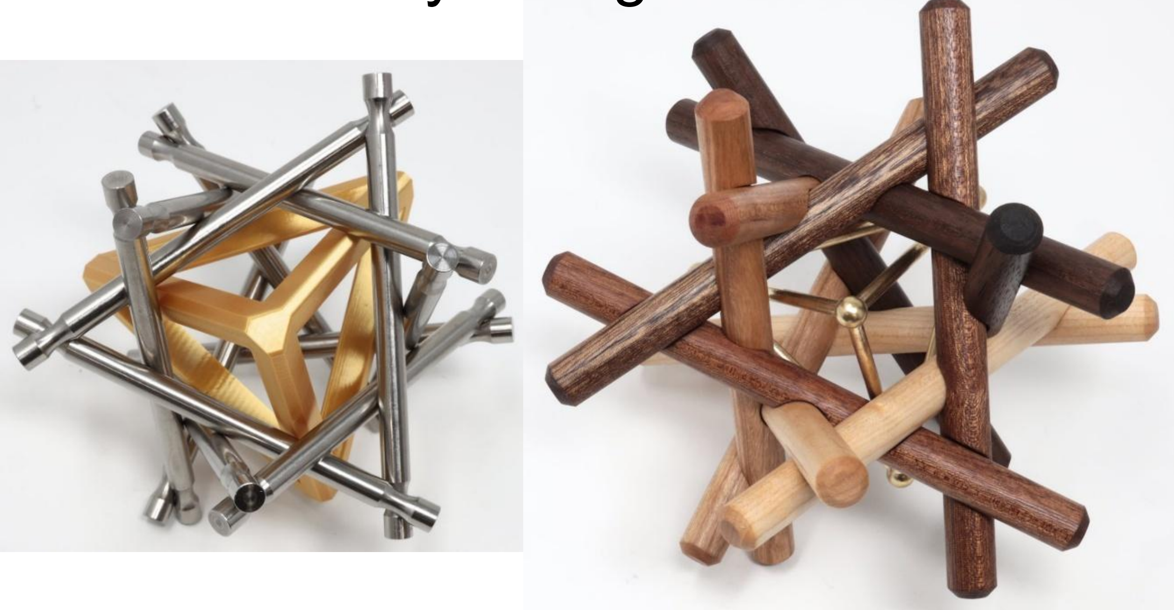

Figure 1. Solved positions of *Nova Plexus* by Geoff Wyvill (made by Two Brass Monkeys [1]) and *Tetra Cage* by George Bell.

Introduction

Nova Plexus [1] by Geoff Wyvill (Figure 1) is one of my favourite puzzles. The puzzle consists of twelve identical cylinders or rods and your goal is to assemble them into a self-supporting structure. Remarkably, when properly tightened, *Nova Plexus* is stable and does not fall apart. The symmetry of the rod arrangement is not obvious but is tetrahedral. The symmetry is easier to see when a tetrahedron is added to the interior.

Part of the appeal of *Nova Plexus* is its simplicity and complete lack of obfuscation. You begin with a pile of 12 smooth metal rods. Your goal of building a self-supporting structure seems laughably absurd—yet it is possible.

In this article we will show one way to derive the shape of Nova Plexus. We then describe a related puzzle obtained from the same configuration of 12 rods which I call *Tetra Cage*. For *Tetra Cage* the rod diameters are doubled, and half-depth notches are added to the pieces. I will discuss my efforts to make *Tetra Cage*, both in wood and using 3D printing.

Symmetric arrangements of twelve rods have been discussed previously in [2–4]. We will find many similarities with this work, the major difference being that for *Nova Plexus* the arrangement of triangles is not concentric (the centre of each triangle is not the centre of the puzzle).

Puzzle geometry

An important detail of *Nova Plexus* is that the metal rods have hyperbolic notches at their ends, which keeps them from sliding off. Initially, we will ignore this detail. For now, consider **Pure Nova Plexus**: a symmetrical arrangement of 12 identical rods with each rod touching five neighbours.

To form *Pure Nova Plexus*, we start with a symmetrical arrangement of three mutually touching cylinders, called a **triangle of rods** (TOR), shown in Figure 2. *Pure Nova Plexus* can be created by interlocking four identical TORs. The angle between rod centre lines is $\phi \approx 60°$, and the vertical tilt of each rod in the side view is β.

Each rod has length *L*, diameter *d* and is displaced away from the centre by a distance ℓ. We can derive an exact result for the required rotation angle β in terms of the cylinder displacement and diameter,

$$\cot\beta = 2\sqrt{3\left(\frac{\ell}{d}\right)^2 - 1}$$

Note that when $(\ell/d)^2 = 1/3$, $\beta = 90°$. This corresponds to three parallel rods stacked in a triangle, as shown in Figure 2c. It is not possible to have $\ell/d < 1/\sqrt{3}$, because the rods must overlap.

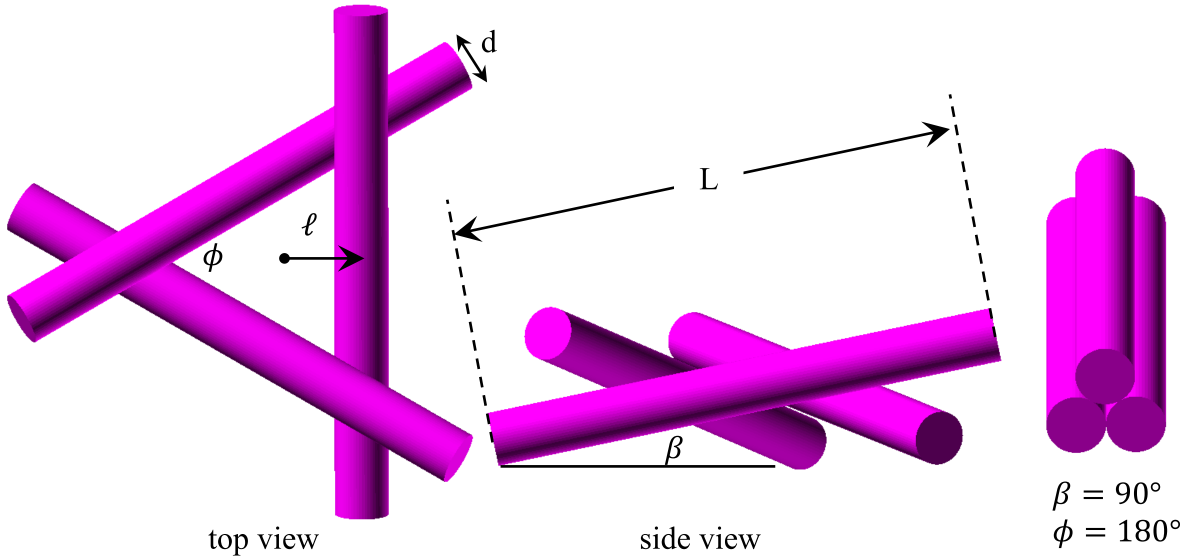

Figure 2. Three views of a triangle of rods (TOR).

As $d \to 0$ the rods become lines, $\beta \to 0$ and $\phi \to 60°$. When $d > 0$, the rods must be tilted and ϕ is greater than 60°. The exact formula for ϕ is

$$\cos\phi = \frac{1}{2} - \frac{1}{8(\ell/d)^2 - 2}.$$

The length of the rods L is arbitrary but must be at least $2\sqrt{3}\ell$, the approximate distance between contact points. By measurement of the *Nova Plexus* in Figure 1, we find that $d = 6$ mm and $\ell/d = 4$. This gives $\beta = 4.1°$, $\phi = 60.52°$ which gives L at least 83 mm. We note that the TOR formulas can be generalized to more than three rods[1], which may be useful in other puzzles.

From this point onward, exact results will not be possible. To form *Pure Nova Plexus*, take four copies of a triangle of rods (TOR) and execute two steps:

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[1] We can generalize to n mutually touching rods, each tilted by β and rotated by $360°/n$.

1) Rotate each TOR around the *z*-axis (out of the page in Figure 2a) by a phase angle, α.
2) Align the axis of each TOR with the vertices of a tetrahedron and translate each away from the tetrahedron centre by an amount D.

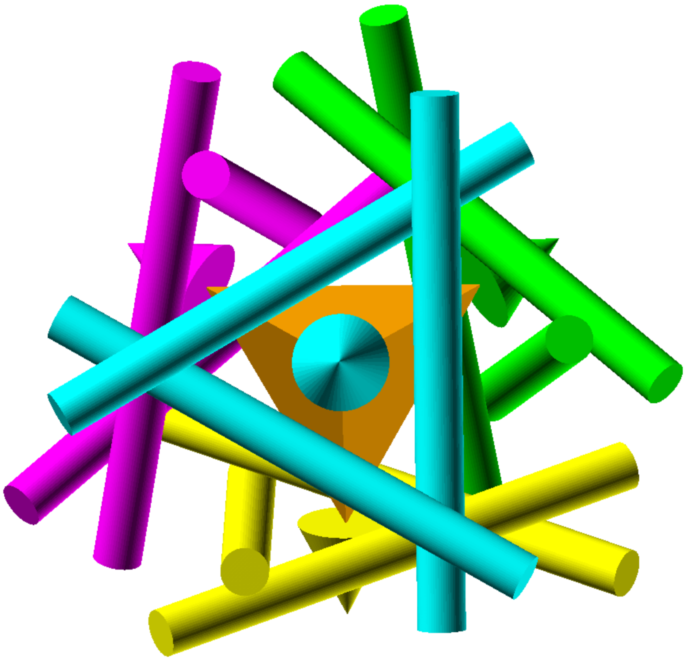

Figure 3. Building Nova Plexus from four TORs.
Notice the similarity to Figure 1.

We note that when $D = 0$ the triangles are concentric (they share the same centre), like the arrangements in [2]. Figure 3 shows four TOR's after step 2 is performed (when $D > 0$ and $\alpha = 0$). The guiding (orange) tetrahedron is also shown.

We now have four free parameters: distances $d,\ \ell,\ D$ and the phase angle α. We can eliminate one parameter by measuring distances in units of rod diameter *d*. Note that each rod always touches two others in its TOR. We now adjust $(\ell/d\,, D/d\,, \alpha)$ so that each rod touches three more.

There doesn't appear to be a formula for a solution, but infinitely many solutions can be found using optimization software. I used a root finding algorithm to find *D* and α starting with $\ell/d = 4$. A solution which appears very close to *Nova Plexus* in Figure 1a is $(\ell/d = 4.0;\ D/d = 2.80832;\ \alpha = -10.52°)$. This becomes our "operating point" in parameter space to create *Tetra Cage*.

Remember that *Nova Plexus* uses rods with hyperbolic notches at each end. Crucially, these notches only affect the TORs. Thus, for *Nova Plexus* we re-run the optimization program using a larger value for the rod separation than was used for the TORs. This will give a slightly different operating point in parameter space.

The *Tetra Cage* puzzle

The idea behind Tetra Cage (Figure 1b) is to double the rod diameter and cut cylindrical notches in the rods to half their depth. There will be one notch for each pair of touching rods and since each of the 12 rods touches 5 others, 30 notches in total. Since 12 doesn't divide 30 evenly, it is not possible to have 12 identical pieces.

We now select a particular operating point and convert all lengths to millimetres. We start with rods of (doubled) diameter 12.7 mm = ½ inch and calculate $\ell = 25.4$ mm, D = 17.833 mm, and $L > 88$ mm. Values for the three angles are $\phi = 60.52°$, $\beta = 4.17°$, $\alpha = -10.52°$.

Figure 4 shows the basic piece with all five notches, top and bottom views, which correspond to looking from inside and outside the puzzle, respectively. At first glance, this piece seems symmetric under 180° rotation about the z-axis (coming out of the page). In CFF 123 [2] I argued that the piece with all notches must have such a symmetry because the symmetry transformation takes the puzzle back to itself. Here, however, when we apply the symmetry transformation to the entire puzzle, it **does not** come back to itself. Thus, **there is no reason to expect the five-notch piece to be symmetric**. In fact, the centre notch (#3) is not in the exact middle of the piece. According to Table 1, this notch is displaced from the centre by one mm!

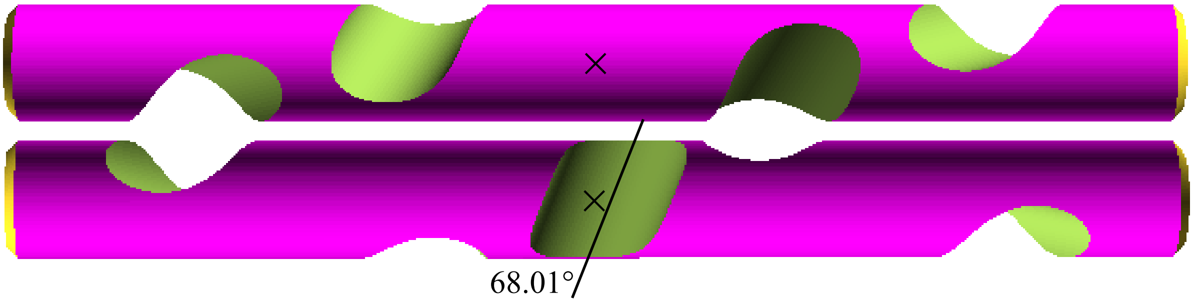

Figure 4. Top and bottom view of the basic five-notch piece. The exact centre of the piece is marked by an ×.

To describe each piece, I list the notches which are present in binary and then convert this to a base 10 number (equivalently, add up the Notch Codes). For example, a useful piece includes four notches: #1, #2, #4 and #5. In binary this is 11011 = 1+2+8+16 = **27** (base 10).

Piece **27** is a useful piece, and visually it appears symmetric under 180° rotation. I assumed for one year that this piece was symmetric—only after calculating Table 1 did I realize notches #2 and #4 are not quite symmetric. In my view, a piece which appears symmetric, but is not, is a problem with the design of any puzzle. The solver may not notice this lack of symmetry and become frustrated when the piece doesn't fit in the wrong orientation.

Notch	Notch Code	Angle	Location (centre)
#1	1	60.52° (β)	43.19 mm
#2	2	68.01°	19.81 mm
#3	4	68.01°	1.03 mm
#4	8	69.65°	-20.16 mm
#5	16	60.52° (β)	-43.19 mm

Table 1. Location and angle for the five notches, relative to the centre (calculated from the mathematical model).

The way puzzle pieces are created, the location of any notch is only accurate to within a parameter known as the **offset**. Thus, if notches #2 and #4 are symmetric to within the offset, piece **27** should fit in either orientation.

The total number of possible pieces is $2^5 = 32$. There are many possible designs, including some where all twelve pieces are different. In playing around with prototypes, I found that the pieces could be added using two types of moves:

1) Pieces inserted axially, i.e. along their length.
2) Pieces which are rotated into position.

The structure is quite unstable initially, and I found moves of the second type to be more satisfying. The design I eventually settled on uses the following twelve pieces: 3×**2**, 3×**27**, 3×**15**, **17**, **1**, **0**.

To assist with the initial stability of *Tetra Cage* rubber bands are useful, as with *Nova Plexus*. The final three pieces are inserted axially. After the final piece is in place the structure is very stable, and the rubber bands can be removed.

3D Printing *Tetra Cage*

Starting from the mathematical model, I used OpenSCAD [5] to generate stl files of the puzzle pieces. The pieces can be printed vertically; a large brim is needed to keep them attached to the build plate. A more serious problem is that in this orientation the finished pieces are weak. Made with 1 mm wall thickness a piece is easily broken at a notch. I increased the wall thickness to 3 mm and have not broken any pieces, but I am careful to limit any bending force. Printing the pieces horizontally or at an angle they are much stronger but supports are needed and accurate printing becomes more difficult.

The advantage of vertically printed pieces is that the notch locations and angles are extremely accurate. I use an offset of 0.1 mm. The plastic version is the only version where one can notice a difference in tightness after flipping the **27** pieces.

Making *Tetra Cage* in wood

It seemed this might not be difficult, as it requires only cutting notches in dowels. However, getting the notches in the correct place proved tricky.

My idea was to use 3D printed jigs to hold each dowel securely, with an alignment hole for drilling of each notch using a drill press. Figure 5 shows one of my jigs. It is constructed from two halves which are screwed together using M4 machine screws with square nuts. When the halves are screwed together tightly, the dowel is held securely in place.

Initially I tried to make one jig to cut all five notches. This was overly ambitious; eventually I settled on three jigs to be used in the following order:

A. For cutting notches #1 and #5
B. For cutting notches #2 and #4
C. For cutting the central notch #3

Jigs A and B are made so that one can simply flip the jig which orients it for the second hole. I securely fix the jig to the drill press rail using a clamp. Jigs B and C contain additional holes for piece alignment. All jigs were created using OpenSCAD [3].

This worked reasonably well, although I was never able to get accuracy down to the level of the 3D printed versions. The best offset I was able to achieve with the wood versions was around 0.3 mm.

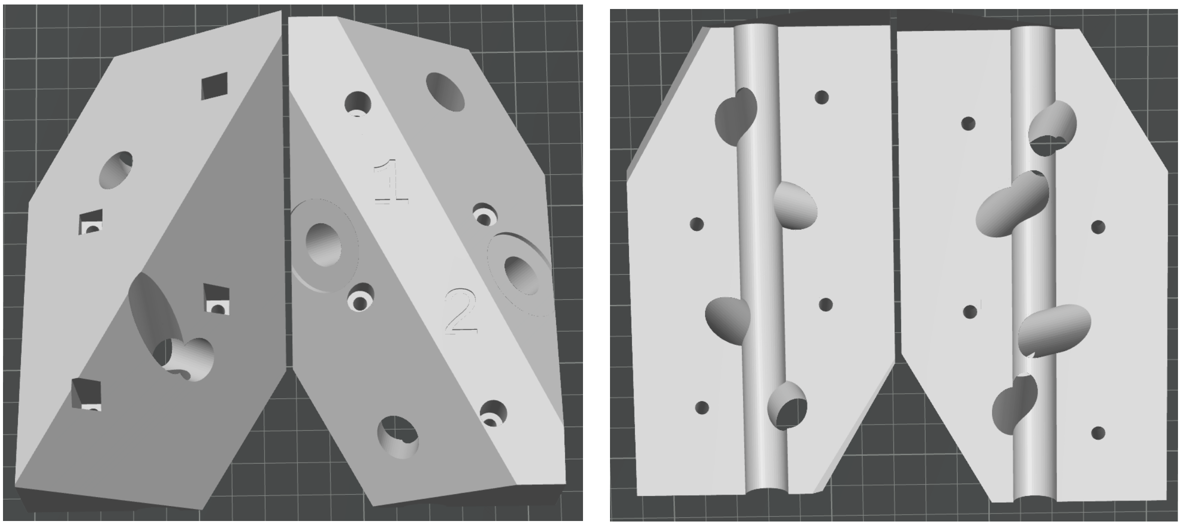

Figure 5. Jig B for making Tetra Cage (top and bottom views of the two halves).

Lessons learned:

1) All drill bits have difficulty making an accurate offset hole because they bend away from the wood. I performed some tests and found a Freud Serrated Edge Forstner bit made the most accurate offset holes.
2) If the dowel diameter was too small, the dowel would spin inside the jigs. About 20% of my dowels were on the small side and could not be used. I tried wrapping them with tape, but this slowed my process down.
3) By Table 1, Notches #2 and #4 are not quite symmetric. Thus, Jig B was not perfectly symmetric which introduced tiny misalignments upon flipping the jig.
4) Many pieces required notches from several jigs, and proper piece alignment in jigs B and C was essential (the extra holes in Figure 5 are for alignment).
5) The biggest problem with my jigs was using them was slow. Cutting 30 notches for one puzzle took me over two hours, and there seemed no way to significantly speed this up. For this reason, only about 10 copies of *Tetra Cage* were made.

Summary

We have created a mathematical model of *Nova Plexus*, a configuration of twelve identical rods where each rod touches five neighbours. Using this model we have created *Tetra Cage*, where the rod diameter is doubled and notches are cut in the pieces. We have shown that the notching pattern is not symmetric under 180° rotation. There are 32 pieces possible and many piece combinations can be assembled into a *Tetra Cage*. *Tetra Cage* may be available in my Etsy store [6].

References

[1] Nova Plexus by Geoff Wyvill, see https://twobrassmonkeys.com/
[2] George Bell, Screwy Burrs, CFF 123, Mar 2024, p. 16-21
[3] Bruce Patterson, Equivalent Rods in Space: Polygon Arrangements, CFF 118, Jul 2022, p. 36–40.
[4] George Hart, Symmetric Stick Puzzles, in Proc. of Bridges 2011, pp. 357–364.
[5] OpenSCAD, https://openscad.org/
[6] PolyPuzzles, https://www.etsy.com/shop/PolyPuzzles